Answer on Question #76441 – Math – Calculus

Question

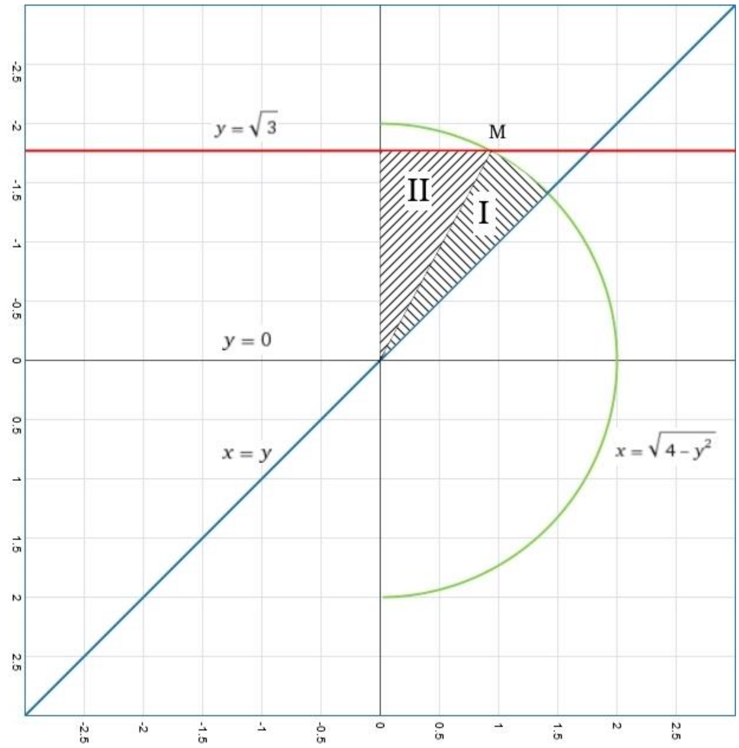
Evaluate the integral by converting to polar coordinates

$$\int_0^{\sqrt{3}} \int_y^{\sqrt{4-y^2}} \frac{dxdy}{4+x^2+y^2};$$

Solution

Double Integrals in Polar Coordinates:

$$\iint_D f(x,y)dxdy = \iint_\Omega f(r\cos\phi, r\sin\phi)\, rdrd\phi;$$



Intersection point M:

$$y=\sqrt{3};$$

$$2\sin\phi = \sqrt{3};\ \ \phi = \frac{\pi}{3}.$$

We have two areas:

I: $0 \le r \le 2;\ \frac{\pi}{4} \le \phi \le \frac{\pi}{3};$

II: $0 \le r \le \frac{\sqrt{3}}{\sin\phi};\ \frac{\pi}{3} \le \phi \le \frac{\pi}{2};$

$$4+x^2+y^2 = 4+(r\sin\phi)^2+(r\cos\phi)^2 = 4+r^2.$$

$$\int_0^{\sqrt{3}} \int_y^{\sqrt{4-y^2}} \frac{dxdy}{4+x^2+y^2} = \int_{\pi/4}^{\pi/3} d\phi \int_0^2 \frac{rdr}{4+r^2} + \int_{\pi/3}^{\pi/2} d\phi \int_0^{\frac{\sqrt{3}}{\sin\phi}} \frac{rdr}{4+r^2} = \left(\frac{\pi}{3}-\frac{\pi}{4}\right)\frac{1}{2}\int_0^2 \frac{d(4+r^2)}{4+r^2} + \int_{\pi/3}^{\pi/2} d\phi \int_0^{\frac{\sqrt{3}}{\sin\phi}} \frac{d(4+r^2)}{4+r^2}$$

$$= \frac{\pi}{24}[\ln(4+r^2)]_0^2 + \int_{\pi/3}^{\pi/2} d\phi [\ln(4+r^2)]_0^{\sqrt{3}/\sin\phi}$$

$$= \frac{\pi}{24}(\ln 8 - \ln 4) + \int_{\pi/3}^{\pi/2} d\phi \left[\ln\left(4+\frac{3}{(\sin\phi)^2}\right) - \ln 4\right] = \frac{\pi \ln 2}{24} + \int_{\pi/3}^{\pi/2} d\phi \ln\left(1+\frac{3}{4(\sin\phi)^2}\right).$$

Last integral is not defined by the elementary functions; its circled value is 0.315155.

Answer: $I = \frac{\pi \ln 2}{24} + 0.315155.$