Answer on Question #76041 – Math – Differential Equations
Question

Find the surface which is orthogonal to the one parameter system

$$z = cxy(x^2 + y^2)$$

and which passes through the hyperbola

$$x^2 - y^2 = a, z = 0$$

Solution

$$f(x, y, z) = \frac{z}{xy(x^2 + y^2)} = c$$

$$\frac{\partial f}{\partial x} = -\frac{z}{y}\frac{3x^2 + y^2}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = -\frac{z}{x}\frac{3y^2 + x^2}{y^2(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial z} = \frac{1}{xy(x^2 + y^2)}$$

Let $z(x, y)$ is a surface orthogonal to the given system. Then:

$$(f_x, f_y, f_z) \cdot (z_x, z_y, -1) = 0$$

So, we have differential equation:

$$-\frac{z}{y}\frac{3x^2 + y^2}{x^2(x^2 + y^2)^2} z_x - \frac{z}{x}\frac{3y^2 + x^2}{y^2(x^2 + y^2)^2} z_y - \frac{1}{xy(x^2 + y^2)} = 0$$

Divide by $-\frac{z}{xy(x^2+y^2)}$:

$$\frac{3x^2 + y^2}{x(x^2 + y^2)} z_x + \frac{3y^2 + x^2}{y(x^2 + y^2)} z_y + \frac{1}{z} = 0$$

The auxiliary equations:

$$\frac{x(x^2 + y^2)dx}{3x^2 + y^2} = \frac{y(x^2 + y^2)dy}{3y^2 + x^2} = -zdz$$

Adding the first and the second equations and equating to the third one:

$$\frac{(x^2+y^2)(xdx+ydy)}{4(x^2+y^2)}=\frac{(xdx+ydy)}{4}=-zdz$$

$$x^2+y^2+4z^2=c_1$$

Subtracting the second equation from the first one:

$$\frac{(x^2+y^2)(xdx-ydy)}{2(x^2-y^2)}=\frac{(c_1-4z^2)(xdx-ydy)}{2(x^2-y^2)}=-zdz$$

$$x^2-y^2=c_2\sqrt{c_1-4z^2}$$

$$c_2=\frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

Using the given conditions $x^2-y^2=a, z=0$:

$$a^2=c_2^2c_1$$

Then:

$$a^2=(x^2+y^2+4z^2)\cdot\frac{(x^2-y^2)^2}{x^2+y^2}$$

$$(x^2+y^2)a^2=(x^2+y^2+4z^2)(x^2-y^2)^2$$

$$z^2=\frac{(x^2+y^2)a^2}{4(x^2-y^2)^2}-\frac{x^2+y^2}{4}$$

Answer:

$$z=\sqrt{\frac{(x^2+y^2)a^2}{4(x^2-y^2)^2}-\frac{x^2+y^2}{4}}$$