Answer on Question #76036 – Math – Differential Equations

Solve the differential equation

Question

1.

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$$

Solution

$$\frac{dy}{dx} = \sin(x + y)$$

$$x + y = u$$

$$1 + \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \sin u$$

$$\frac{du}{dx} = \sin u + 1$$

$$\int \frac{du}{\sin u + 1} = \int dx = x + c$$

$$\int \frac{du}{\sin u + 1} = \int \frac{1 - \sin u}{1 - sin^2 u} du = \int \frac{du}{cos^2 u} - \int \frac{\sin u}{cos^2 u} du$$

$$\int \frac{\sin u}{cos^2 u} du = -\int \frac{d(\cos u)}{cos^2 u} = \frac{1}{\cos u}$$

$$\int \frac{du}{cos^2 u} = \int (1 + tan^2 u) du \Rightarrow t = \tan u\,, dt = (1 + t^2) du, du = \frac{dt}{1 + t^2}$$

$$\int \frac{du}{cos^2 u} = \int \frac{1 + t^2}{1 + t^2} dt = t = \tan u$$

$$\tan u - \frac{1}{\cos u} = x + c$$

Answer:

$$\tan(x+y)-\frac{1}{\cos(x+y)}=x+c$$

Question

2.

$$(1+y^2)dx=(\tan^{-1}y-x)dy$$

Solution

$$\frac{dx}{dy}=\frac{\tan^{-1}y-x}{1+y^2}$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

Integrating factor:

$$e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

$$xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy$$

$$t=\tan^{-1}y$$

$$dt=\frac{1}{1+y^2}dy$$

$$xe^{\tan^{-1}y}=\int te^{t}dt=te^{t}-e^{t}+c=e^{t}(t-1)+c$$

$$xe^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$$

Answer:

$$x=\tan^{-1}y-1+ce^{-\tan^{-1}y}$$

Question

3.

$$(D-1)^2(D^2+1)^2y = \left(\sin\frac{x}{2}\right)^2 + e^x + x$$

Solution

$$\left(\sin\frac{x}{2}\right)^2 = \frac{1-\cos x}{2}$$

$$(D-1)^2(D^2+1)^2y = \frac{1-\cos x}{2} + e^x + x$$

$$(y'-y)^2(y''+y)^2 = 0$$

Auxiliary equation:

$$(\lambda-1)^2(\lambda^2+1)^2 = 0$$

$$\lambda_{1,2} = 1, \lambda_{3,4} = i, \lambda_{5,6} = -i$$

General solution:

$$Y = c_1e^x + c_2xe^x + c_3\cos x + c_4\sin x + c_5x\cos x + c_6x\sin x$$

Particular integral:

$$\tilde{y} = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3$$

Particular integral $\tilde{y}_1$ for:

$$(y'-y)^2(y''+y)^2 = \frac{1}{2} + x$$

$$\tilde{y}_1 = A + Bx$$

$$\tilde{y}_1{}' = B$$

$$\tilde{y}_1{}'' = 0$$

$$(B-A-Bx)^2(A+Bx)^2 = \frac{1}{2} + x$$

We cannot find coefficients A, B using analytical methods.

Particular integral $\tilde{y}_2$ for:

$$(y' - y)^2(y'' + y)^2 = e^x$$

$$\tilde{y}_2 = Ax^2e^x$$

$$\tilde{y}_2{}' = 2Axe^x + Ax^2e^x$$

$$\tilde{y}_2{}'' = 2Axe^x + 2Ae^x + 2Axe^x + Ax^2e^x = 2Ae^x + 4Axe^x + Ax^2e^x$$

$$(2Axe^x + Ax^2e^x - Ax^2e^x)^2(2Ae^x + 4Axe^x + Ax^2e^x + Ax^2e^x)^2 = e^x$$

$$4A^2x^2e^{2x}(2Ae^x + 4Axe^x + 2Ax^2e^x)^2 = e^x$$

We cannot find coefficient A using analytical methods.

Particular integral $\tilde{y}_3$ for:

$$(y' - y)^2(y'' + y)^2 = -\frac{1}{2}\cos x$$

$$\tilde{y}_3 = Ax^2\cos x + Bx^2\sin x$$

$$\tilde{y}_3' = 2x(A\cos x + B\sin x) + x^2(B\cos x - A\sin x)$$

$$\tilde{y}_3'' = 2(A\cos x + B\sin x) + 4x(B\cos x - A\sin x) - x^2(A\cos x + B\sin x)$$

$$\left((A\cos x + B\sin x)(2x - x^2) + x^2(B\cos x - A\sin x)\right)^2\left(2(A\cos x + B\sin x) + 4x(B\cos x - A\sin x)\right)^2 = \\ = -\frac{1}{2}\cos x$$

We cannot find coefficients A, B using analytical methods.

Answer: It is impossible to solve the given equation using analytical methods, and for numerical methods we have not initial value.
We can use, for example, Runge-Kutta method (if we have initial value) which uses iterations to find value $y(x)$ on the defined interval.

Question

4.

$$2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

Solution

$$u=\frac{y_x}{y}$$

$$u_x+\frac{u^2}{2}-\frac{u}{x}+\frac{2}{x^2}=0$$

$$v(x)=e^{\frac{1}{2}\int u(x)dx}$$

$$x^2v_{xx}-xv_x+v=0$$

$$v(x)=|x|(c_1+c_2\ln|x|)$$

$$\ln v=\frac{1}{2}\int u(x)dx$$

$$\frac{v_x}{v}=\frac{u}{2}$$

$$v_x=c_1+c_2+c_2\ln x$$

$$\frac{c_1+c_2+c_2\ln x}{x(c_1+c_2\ln x)}=\frac{y_x}{2y}$$

$$\frac{c_1+c_2+c_2\ln x}{x(c_1+c_2\ln x)}dx=\frac{dy}{2y}$$

$$\int\frac{c_1+c_2+c_2\ln x}{x(c_1+c_2\ln x)}dx=\int\frac{dy}{2y}$$

Answer:

$$y=c_2x^2(c_1+\ln x)^2$$