Answer on Question #76002 – Math – Differential Equations

Question

Obtain a solution of the wave equation
∂^2u(x,t)/∂t^2=16(∂^2u(x,t)/∂x^2)
for 0≤ x ≤ π and t > 0 and the following boundary and initial conditions:
u(0,t)=u(π,t)=0,
u(x,0)=x(π-x) and ∂u(x,0)/∂t=0

Solution

We consider a one-dimensional homogeneous wave equation on the interval [0, π]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad c = 4$$

boundary conditions:

$$u(0,t) = u(\pi,t) = 0$$

and initial conditions:

$$u(x,0) = x(\pi - x), \quad \frac{\partial u(x,0)}{\partial t} = 0$$

Equation can be solved exactly by d'Alembert's formula, using a Fourier transform method, or via separation of variables:

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{l}x\right)\left[\alpha_k \cos\left(\frac{ck\pi}{l}t\right) + \beta_k \sin\left(\frac{ck\pi}{l}t\right)\right] = \{l = \pi, c = 4\}$$
$$= \sum_{k=1}^{\infty} \sin(kx)\,[\alpha_k \cos(4kt) + \beta_k \sin(4kt)]$$

with:

$$\beta_k = \frac{2}{ck\pi}\int_0^l \frac{\partial u(x,0)}{\partial t}\,\sin\left(\frac{k\pi}{l}x\right)dx = \left\{\frac{\partial u(x,0)}{\partial t} = 0\right\} = 0$$

$$\alpha_k = \frac{2}{l}\int_0^l u(x,0)\,\sin\left(\frac{k\pi}{l}x\right)dx = \{l = \pi \ \ and \ \ u(x,0) = x(\pi - x)\}$$
$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,\sin(kx)\,dx$$
$$= \frac{2}{\pi}\left(\pi\int_0^{\pi} x\,\sin(kx)\,dx - \int_0^{\pi} x^2\,\sin(kx)\,dx\right)$$

Calculate $\int_0^{\pi} x\,\sin(kx)\,dx$:

$$\int_0^{\pi} x\ \sin(kx)\,dx = \left\{\int v\,du = v\,u - \int u\,dv \quad let\ v = x\ \ and\ \ du = \sin(kx)\,dx\right\}$$
$$= \frac{-x}{k}\cos(kx)\ |_0^{\pi} + \int_0^{\pi} \frac{\cos(kx)}{k}\,dx = \frac{-x}{k}\cos(kx)\ |_0^{\pi} + \frac{1}{k^2}\sin(kx)\ |_0^{\pi}$$
$$= \frac{\sin(\pi k) - \pi k\cos(\pi k)}{k^2}$$

Calculate $\int_0^{\pi} x^2\ \sin(kx)\,dx$:

$$\int_0^{\pi} x^2\ \sin(kx)\,dx = \left\{\int v\,du = v\,u - \int u\,dv \quad let\ v = x^2\ \ and\ \ du = \sin(kx)\,dx\right\}$$
$$= \frac{-x^2}{k}\cos(kx)\ |_0^{\pi} + \frac{2}{k}\int_0^{\pi} x\cos(kx)\,dx$$
$$= \left\{\int v\,du = v\,u - \int u\,dv \quad let\ v = x\ \ and\ \ du = \cos(kx)\,dx\right\}$$
$$= \frac{-x^2}{k}\cos(kx)\ |_0^{\pi} + \frac{2}{k}\left(\frac{x}{k}\sin(kx)\ |_0^{\pi} - \int_0^{\pi}\frac{\sin(kx)}{k}\,dx\right)$$
$$= \left(\frac{2}{k^3} - \frac{x^2}{k}\right)\cos(kx)\ |_0^{\pi} + \frac{2x}{k^2}\sin(kx)\ |_0^{\pi}$$
$$= \frac{2 - (\pi k)^2}{k^3}\cos(\pi k) - \frac{2}{k^3} + \frac{2\pi}{k^2}\sin(\pi k)$$

Then

$$\alpha_k = \frac{2}{\pi}\left(\pi k\,\frac{\sin(\pi k) - \pi k\cos(\pi k)}{k^3} - \left[\frac{2 - (\pi k)^2}{k^3}\cos(\pi k) - \frac{2}{k^3} + \frac{2\pi k}{k^3}\sin(\pi k)\right]\right)$$
$$= \frac{2}{\pi}\left(-\pi k\,\frac{\sin(\pi k)}{k^3} + \frac{2}{k^3}(1 - \cos(\pi k))\right)$$
$$= \frac{2}{\pi}\left(\frac{-\pi k\ \sin(\pi k) + 4\,(\sin(\pi k/2))^2}{k^3}\right)$$
$$= \{\sin(\pi k) = 0 \quad and \quad (\sin(\pi k/2))^2 = 1 \ \ for\ \ k = 1,2\ldots\} = \frac{8}{\pi\cdot k^3}$$

and

$$u(x,t) = \sum_{k=1}^{\infty}\alpha_k\cos(4kt)\ \sin(kx) = \sum_{k=1}^{\infty}\frac{8}{\pi\cdot k^3}\cos(4kt)\ \sin(kx)$$

Answer: $u(x,t) = \sum_{k=1}^{\infty}\frac{8}{\pi\cdot k^3}\cos(4kt)\sin(kx)$

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