ANSWER on Question #75681 – Math – Calculus

QUESTION

Trace the curve

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$

State the properties used for tracing it.

SOLUTION

We use the following properties for tracing the curve.

We have the Cartesian curve defined by the parametric equations

$$l : \begin{cases} x = f(\theta) \\ y = g(\theta) \end{cases}$$

Since $y = g(\theta)$ is a periodic function of θ with period 2π, it is sufficient to trace the curve for $\theta \in [0; 2\pi]$.

For $\theta \in [0; 2\pi]$, x and y are well defined.

Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta); 0 \leq \theta \leq 2\pi; a > 0$.

1. Symmetry

$$\begin{cases} x = f(\theta) = a(\theta - \sin\theta) \\ y = g(\theta) = a(1 - \cos\theta) \end{cases}$$

$$f(-\theta) = a(-\theta - \sin(-\theta)) = a(-\theta - [-\sin\theta]) = a(-\theta + \sin\theta) = -a(\theta - \sin\theta) = f(\theta)$$

$$\boxed{f(-\theta) = f(\theta)}$$

$$g(-\theta) = a(1 - \cos(-\theta)) = a(1 - \cos\theta) = g(\theta)$$

$$\boxed{g(-\theta) = g(\theta)}$$

Therefore, the curve is symmetrical about the y −axis.

Curve is not symmetrical about x −axis.

Curve is not symmetrical about the line $y = x$.

Curve is not symmetrical about the line $y = -x$.

Curve is not symmetrical in opposite quadrants.

2. Origin (0,0)

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \to \begin{cases} a(\theta - \sin\theta) = 0 \\ a(1 - \cos\theta) = 0 \end{cases} \to \begin{cases} \theta - \sin\theta = 0 \\ 1 - \cos\theta = 0 \end{cases} \to \begin{cases} \sin\theta = \theta \\ \cos\theta = 1 \end{cases} \to \begin{cases} \sin\theta = \theta \\ \theta = 0 \;\; or \;\; \theta = 2\pi \end{cases} \to$$

$$\theta = 0$$

Conclusion,

$$\boxed{A\ curve\ passes\ through\ the\ origin}$$

3. Intercepts

Intersection with x −axis: The points of intersection of the curve with the x −axis are given by the roots of

$$a(1 - \cos\theta) = 0, 0 \le \theta \le 2\pi; a > 0$$

$$1 - \cos\theta = 0 \to \cos\theta = 1 \to \left[\begin{array}{l} \theta = 0 \\ \theta = 2\pi \end{array}\right.$$

We find the x −coordinates of the intersection points

$$\theta = 0 \to x = a(0 - \sin 0) = a \cdot 0 = 0$$

$$\boxed{(0,0) - first\ intersection\ point}$$

$$\theta = 2\pi \to x = a(2\pi - \sin(2\pi)) = a \cdot 2\pi = 2\pi a$$

$$\boxed{(2\pi a, 0) - second\ intersection\ point}$$

Intersection with y −axis: The points of intersection of the curve with the y −axis are given by the roots of

$$a(\theta - \sin\theta) = 0, 0 \le \theta \le 2\pi; a > 0$$

$$\theta - \sin\theta = 0 \to \sin\theta = \theta \to \boxed{\theta = 0}$$

We find the y −coordinates of the intersection points

$$\theta = 0 \to y = a(1 - \cos 0) = a \cdot (1 - 1) = a \cdot 0 = 0$$

$$\boxed{(0,0) - first\ intersection\ point}$$

4. Asymptotes

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases} - is\ well\ defined\ functions\ for\ all\ \theta \in [0,2\pi]$$

There is no vertical asymptote.

There is no horizontal asymptote.

There is no oblique asymptote.

5. Regions where no Part of the curve lies

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$

Note that $y \ge 0$. Entire curve lies above the x −axis $(0 \le y \le 2a)$.

6. First derivative

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cdot \sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)$$

At $\theta = 0 \rightarrow {}^{dy}/_{dx} = \infty$. Tangent to the curve is perpendicular to x −axis at $\theta = 0$.

At $\theta = \pi \rightarrow {}^{dy}/_{dx} = 0$. Tangent to the curve is parallel to x −axis at $\theta = \pi$.

At $\theta = 2\pi \rightarrow {}^{dy}/_{dx} = \infty$. Tangent to the curve is perpendicular to x −axis at $\theta = 2\pi$.

For $0 < \theta < \pi$, ${}^{dy}/_{dx} > 0$.

Therefore, the function $y(x)$ is increasing in this interval.

For $\pi < \theta < 2\pi$, ${}^{dy}/_{dx} < 0$.

Therefore, the function $y(x)$ is decreasing in this interval.

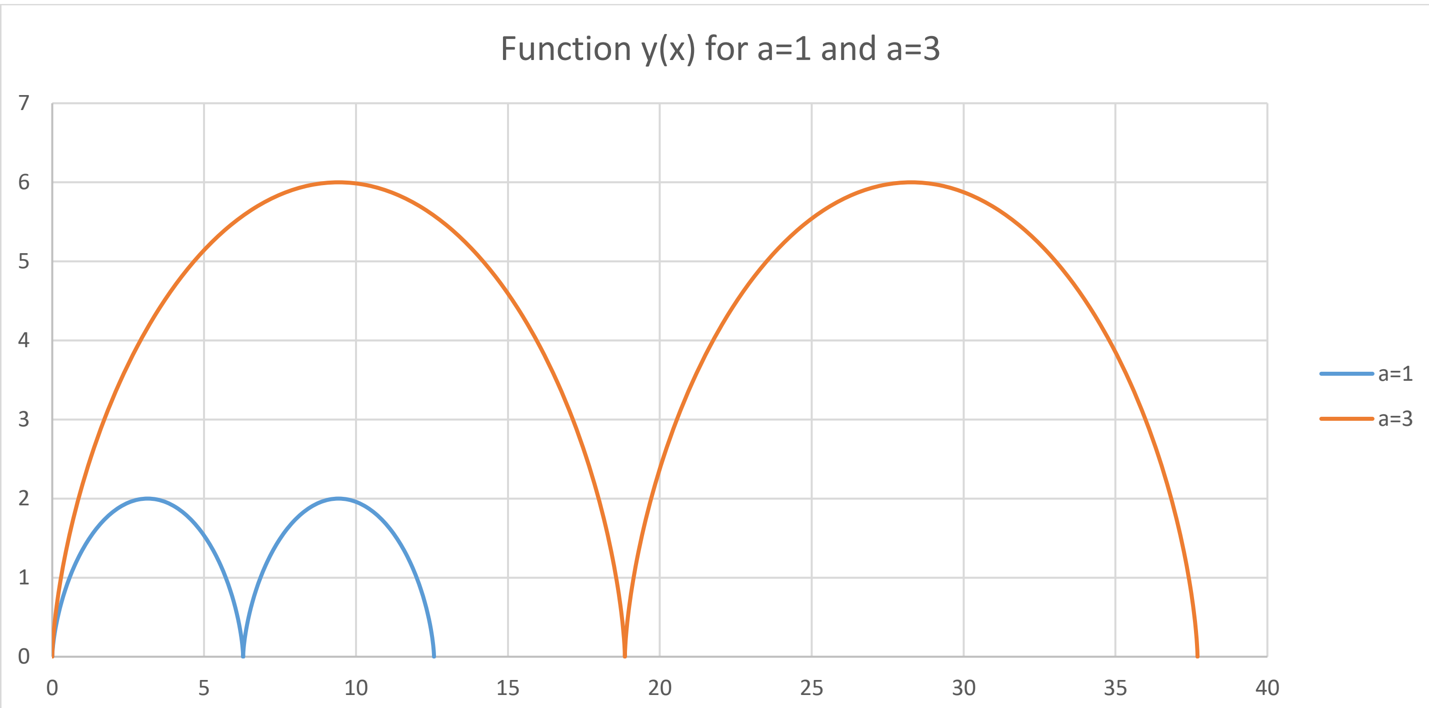
7. Second derivative.

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \\ a > 0 \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)}{\frac{dx}{d\theta}} = \frac{-\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}}{a(1 - \cos\theta)} = \frac{-\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}}{a \cdot 2\sin^2\left(\frac{\theta}{2}\right)} = -\frac{1}{4a\sin^4\left(\frac{\theta}{2}\right)} < 0$$

For $0 < \theta < 2\pi$, ${}^{d^2y}/_{dx^2} < 0 \rightarrow$ concave downward.

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2 - 1)$	$a\pi$	$a(3\pi/2 + 1)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	∞



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