Answer on Question #75451, Math / Algebra

Let x_i belongs to R such that $0 < x_1 \leq x_2 \leq \cdots \leq x_n, n \geq 2$ and

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1.$$

Then show that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right)$$

Solution

Let $a_i = \dfrac{1}{1+x_i}$, for $i = 1, 2, \ldots, n$.

$$1 + x_i = \frac{1}{a_i}, \quad x_i = \frac{1-a_i}{a_i}$$

Then $\sum\limits_{i=1}^{n} a_i = 1$ and the given inequality becomes

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \geq (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} + \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$

$$\sqrt{\frac{1-a_i}{a_i}} + \sqrt{\frac{a_i}{1-a_i}} = \sqrt{\frac{1}{a_i(1-a_i)}}(1 - a_i + a_i) = \sqrt{\frac{1}{a_i(1-a_i)}}$$

Hence

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \geq (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} + \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}} \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> 1 \cdot \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right) \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right) \geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$

Apply the Chebyshev's inequality to the sequences

$(a_1, a_2, \dots, a_n)$ and $\left(\sqrt{\frac{1}{a_1(1-a_1)}}, \sqrt{\frac{1}{a_2(1-a_2)}}, \dots, \sqrt{\frac{1}{a_n(1-a_n)}}\right)$

If $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_i < 1$, for $i = 1, 2, \dots, n$, then

$$\sqrt{\frac{1}{a_1(1-a_1)}} \leq \sqrt{\frac{1}{a_2(1-a_2)}} \leq \cdots \leq \sqrt{\frac{1}{a_n(1-a_n)}}$$

Therefore,

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right) \geq n \sum_{i=1}^{n} (a_i) \left(\sqrt{\frac{1}{a_i(1-a_i)}}\right) <=>$$

$$<=> \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right) \geq n \sum_{i=1}^{n} \left(\sqrt{\frac{a_i^2}{a_i(1-a_i)}}\right) <=>$$

$$<=> \left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right) \geq n \sum_{i=1}^{n} \left(\sqrt{\frac{a_i}{1-a_i}}\right), \text{True}$$

Therefore,

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \geq (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$

Therefore,

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right)$$