Answer on Question #75316 – Math – Differential Equations

Question

A block of mass 1 kg is attached to a spring of force constant k = 25/4 N/m. It is pulled x = 0.3 m from its equilibrium position and released from rest. This spring-block apparatus is submerged in a viscous fluid medium which exerts a damping force of – 4 v (where v is the instantaneous velocity of the block). Determine of the position x(t) of the block at time t.

Solution:

On a block of mass 1 kg, there are three forces:

$$The\ force\ due\ to\ gravity:\ F_g = m \cdot g = 1 \cdot x''$$

$$g = 9.8\, m/s^2\ -\ gravitational\ acceleration$$

$$The\ force\ of\ elasticity\ of\ a\ spring:\quad F_s = -k \cdot x$$

$$k = 25/4\ \frac{N}{m}\ -\ spring\ modulus$$

$$The\ damping\ force:\quad F_d = -d \cdot v = -4 \cdot v = -4 \cdot x'$$

$$\mathrm{d} = 4\ -\ \text{damping coefficient}$$

According to Newton's second law, taking into account the signs of the projections of forces, we obtain the following differential equation with initial conditions:

$$x'' + 4x' + \frac{25}{4}x = 0 \quad IVP:\ x(0) = 0.3, \qquad x'(0) = 0$$

Assuming that the solution has the form $a \cdot e^{r \cdot t}$ we get $(m \cdot r^2 + d \cdot r + k) \cdot a \cdot e^{r \cdot t} = 0$

Upon solving for the roots of the characteristic equation we get the following:

$$r_{1,2} = \frac{-d \pm \sqrt{d^2 - 4mk}}{2m} = \frac{-4 \pm \sqrt{16 - 25}}{2} = -2 \pm \frac{3}{2}i$$

so the displacement is:

$$x(t) = C_1 e^{-2 \cdot t} \cos(3t) + C_2 e^{-2 \cdot t} \sin(3t)$$

$$x'(t) = C_1(-2 \cdot e^{-2 \cdot t} \cos(3t) - 3 \cdot e^{-2 \cdot t} \sin(3t)) + C_2(3 \cdot e^{-2 \cdot t} \cos(3t) - 2 \cdot e^{-2 \cdot t} \sin(3t))$$

We use the initial conditions for finding constants C1 and C2:

$$\begin{cases} C_1 e^{-2 \cdot 0} \cos(0) + C_2 e^{-2 \cdot 0} \sin(0) = 0.3 \\ C_1(-2 \cdot e^{-2 \cdot 0} \cos(0) - 3 \cdot e^{-2 \cdot 0} \sin(0)) + C_2(3 \cdot e^{-2 \cdot 0} \cos(0) - 2 \cdot e^{-2 \cdot 0} \sin(0)) = 0 \end{cases}$$

$$\begin{cases} \mathrm{C}_1 = 0.3 \\ -2\mathrm{C}_1 + 3\mathrm{C}_2 = 0 \end{cases} \quad \rightarrow \quad \mathrm{C}_1 = 0.3, \mathrm{C}_2 = 0.2$$

Then the position x(t) of the block at time t:

$$x(t) = 0.3e^{-2 \cdot t} \cos(3t) + 0.2e^{-2 \cdot t} \sin(3t)$$

Answer: $x(t) = 0.3e^{-2 \cdot t} \cos(3t) + 0.2e^{-2 \cdot t} \sin(3t)$