Answer on Question #75304 – Math – Differential Equations
Question

Solve the following ODE using power series method.

$$(x+2)y''+xy'-y=0$$

Solution

Let assume the solution in form $y=\sum_{n=0}^{\infty}a_nx^n$, exists

Then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

Substitute

$$(x+2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+x\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=1}^{\infty}na_nx^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{N=0}^{\infty}2(N+2)(N+1)a_{N+2}x^N+\sum_{N=1}^{\infty}N(N+1)a_{N+1}x^N+\sum_{N=1}^{\infty}Na_Nx^N-\sum_{N=0}^{\infty}a_Nx^N=0$$

$$2(0+2)(0+1)a_{0+2}x^0+\sum_{N=1}^{\infty}2(N+2)(N+1)a_{N+2}x^N+$$

$$+\sum_{N=1}^{\infty}N(N+1)a_{N+1}x^N+\sum_{N=1}^{\infty}Na_Nx^N-c_0x^0-\sum_{N=1}^{\infty}a_Nx^N=0$$

$$4a_2-a_0=0$$

$$\sum_{N=1}^{\infty}x^N[2(N+2)(N+1)a_{N+2}+N(N+1)a_{N+1}+(N-1)a_N]=0$$

$$2(N+2)(N+1)a_{N+2}+N(N+1)a_{N+1}+(N-1)a_N=0, N\geq 1$$

$$a_{N+2}=-\frac{N(N+1)a_{N+1}+(N-1)a_N}{2(N+2)(N+1)}$$

Plug in $a_0=1, a_1=0$ to find the first solution $y_1(x)$

$$4a_2 - 1 = 0 => a_2 = \frac{1}{4}$$

$$a_3 = -\frac{1(1+1)\left(\frac{1}{4}\right) + (1-1)(0)}{2(1+2)(1+1)} = -\frac{1}{24}$$

$$a_4 = -\frac{2(2+1)\left(-\frac{1}{24}\right) + (2-1)\left(\frac{1}{4}\right)}{2(2+2)(2+1)} = 0$$

$$a_5 = -\frac{3(3+1)(0) + (3-1)\left(-\frac{1}{24}\right)}{2(3+2)(3+1)} = \frac{1}{480}$$

$$a_6 = -\frac{4(4+1)\left(\frac{1}{480}\right) + (4-1)(0)}{4(4+2)(4+1)} = -\frac{1}{2880}$$

$$y_1(x) = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{2880}x^6 + \cdots$$

Plug in $a_0 = 0, a_1 = 1$ to find the second solution $y_2(x)$

$$4a_2 - a_0 = 0 => a_2 = 0$$

$$a_3 = -\frac{1(1+1)(0) + (1-1)(1)}{2(1+2)(1+1)} = 0$$

$$a_4 = -\frac{2(2+1)(0) + (2-1)(0)}{2(2+2)(2+1)} = 0$$

$$a_5 = -\frac{3(3+1)(0) + (3-1)(0)}{2(3+2)(3+1)} = 0$$

$$a_6 = -\frac{4(4+1)(0) + (4-1)(0)}{4(4+2)(4+1)} = 0$$

$$y_2(x) = x$$

The general solution is

$$y = c_1 y_1 + c_2 x$$

Answer: $y = c_1 y_1 + c_2 x$,

$$y_1(x) = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \frac{1}{2880}x^6 + \cdots$$

$$y_2(x) = x$$