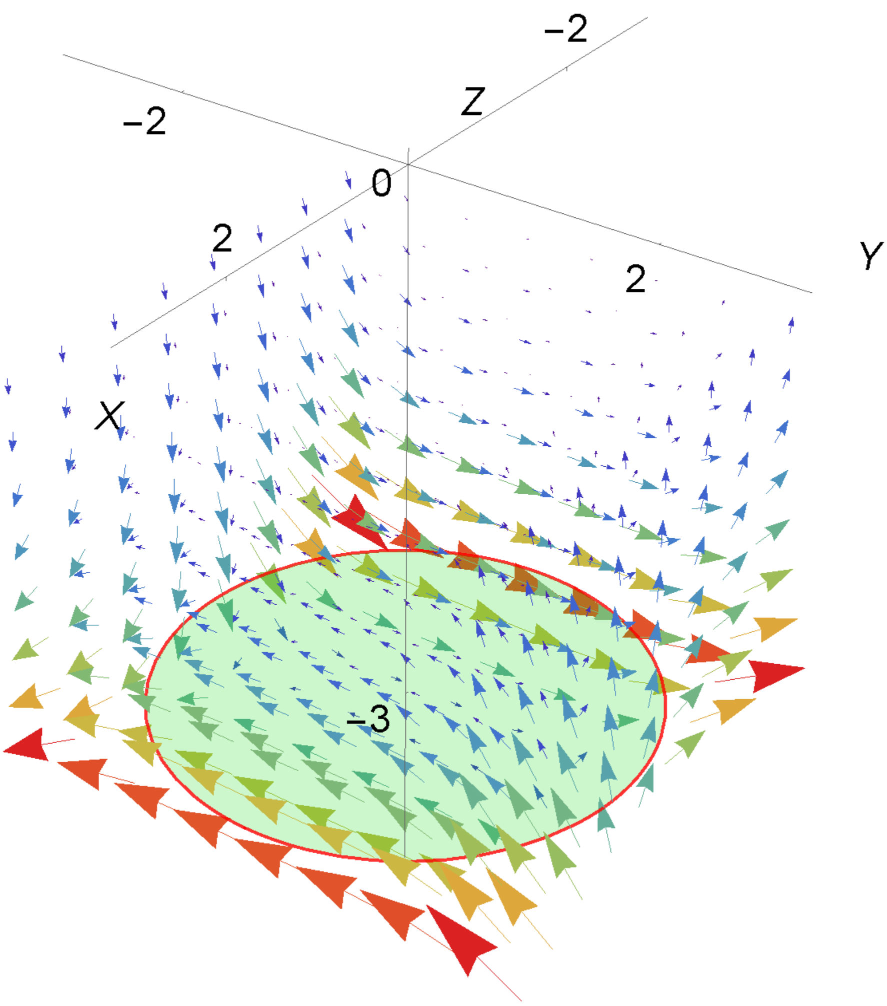
Answer on Question #75295 – Math – Calculus

Question

Using Stokes' Theorem evaluate the line integral ∫ F.dl where F = < $y, xz^3, -zy^3$ >

and C is a circle $x^2 + y^2 = 4$, in the plane z = – 3?

Solution



Stokes' Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = \left(\frac{\partial(-zy^3)}{\partial y} - \frac{\partial(xz^3)}{\partial z}\right)\vec{i} - \left(\frac{\partial(-zy^3)}{\partial x} - \frac{\partial y}{\partial z}\right)\vec{j} + \left(\frac{\partial(xz^3)}{\partial x} - \frac{\partial y}{\partial y}\right)\vec{k}$$
$$= \langle -3zy^2 - 3xz^2, 0, z^3 - 1\rangle$$

$$d\vec{S} = \vec{n}\, dA$$

$$\vec{r} = \langle x, y, -3\rangle \to \vec{r_x} = \langle 1,0,0\rangle, \vec{r_y} = \langle 0,1,0\rangle \to \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1\rangle \to \vec{n} = \frac{\vec{r_x}\times\vec{r_y}}{\|\vec{r_x}\times\vec{r_y}\|}$$
$$= \langle 0,0,1\rangle$$

the surface S be the part of the plane z=-3 bounded by the circle. Obviously, that the unit normal vector is $\vec{n} = \vec{k} = \langle 0,0,1\rangle$

$$\nabla\times\vec{F}\cdot\vec{n} = (-3zy^2 - 3xz^2)\cdot 0 + 0\cdot 0 + (z^3 - 1)\cdot 1 = z^3 - 1$$

$$x = R\cos\theta, y = R\sin\theta, z = -3, dA = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{vmatrix} dR\, d\theta = R\, dR\, d\theta$$

$$x^2 + y^2 \le 4 \;\to\; 0 \le R \le 2, \qquad 0 \le \theta \le 2\pi$$

So

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S (z^3 - 1)\, dA = -28\int_0^{2\pi}\int_0^2 R\, dR\, d\theta = -28\cdot 4\pi = -112\pi$$

Answer: $\oint_C \vec{F}\cdot d\vec{r} = -112\pi.$