Task. Let X be a finite set with $|X| > 1$. What is the difference between $P_1 = X \times X$ and $P_2 = \{S \in P(X) : |S| = 2\}$? Which set, P_1 or P_2, has more elements?

Solution. So,

$$P_1 = X \times X = \{(a,b) : a, b \in X\},$$

$$P_2 = \{S \in P(X) : |S| = 2\} = \{\{a,b\} : a, b \in X, a \neq b\}.$$

Example,

if $a \in X$ then $(a,a) \in P_1$ but $\{a,a\} \notin P_2$;

if $a, b \in X$, then $(a,b), (b,a) \in P_1$ (two elements) and $\{a,b\} = \{b,a\} \in P_2$ (one element).

Therefore, the set P_1 has more elements than the set P_2.

More detail, let $|X| = n$. So,

$$|P_2| = \binom{n}{2} = \frac{n!}{2! \cdot (n-2)!} = \frac{n(n-1)}{2} \text{ and } |P_1| = |X \times X| = |X|^2 = n^2.$$

Then $|P_1| - |P_2| = n^2 - \frac{n(n-1)}{2} = \frac{2n^2 - n(n-1)}{2} = \frac{2n^2 - n^2 + n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$.