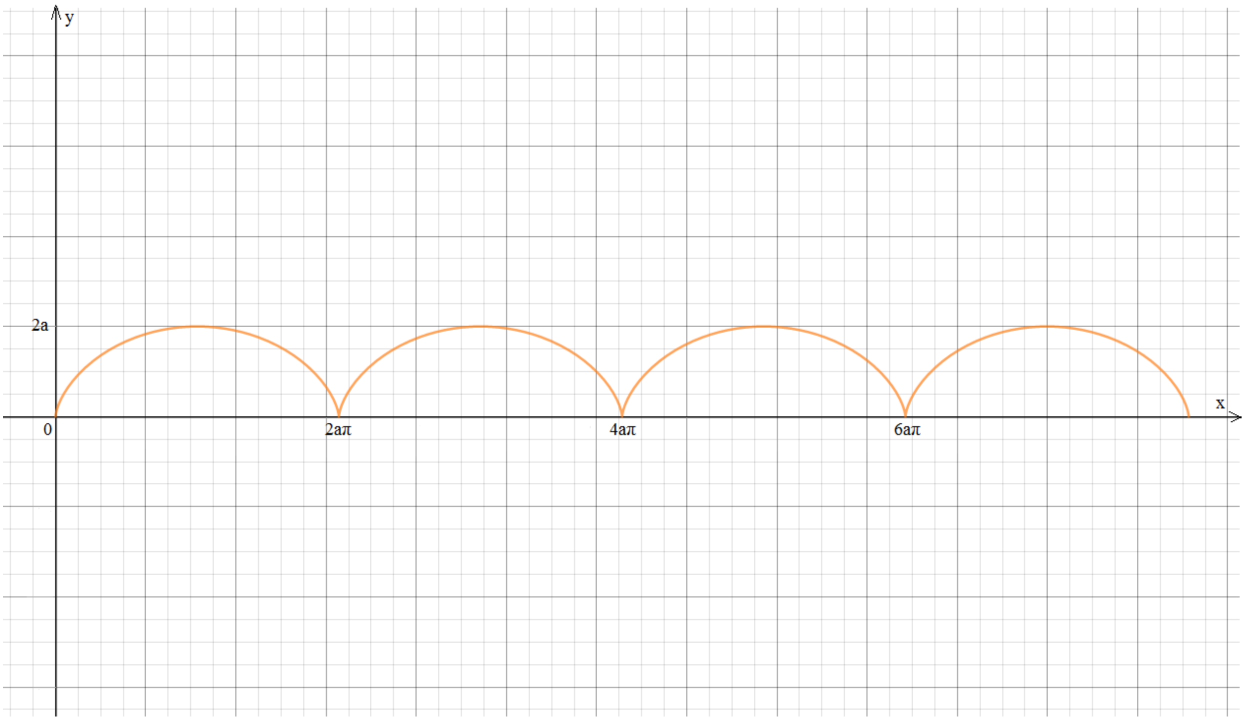
Answer on Question #74502 – Math – Calculus

Question

Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$. State the properties you use for tracing it also.

Solution



We use the following properties for tracing the curve.
We have the Cartesian curve defined by the parametric equations $x = f(\theta)$, $y = g(\theta)$. Since y is a periodic function of θ with period 2π, it is sufficient to trace the curve for $\theta \in [0, 2\pi]$.
For $\theta \in [0, 2\pi], x$ and y are well defined.
Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta); 0 \le \theta \le 2\pi; a > 0$.

1. Symmetry
$x = f(\theta) = a(\theta - \sin\theta);$
$y = g(\theta) = a(1 - \cos\theta)$

$f(-\theta) = a(-\theta - \sin(-\theta)) = -f(\theta);$
$g(-\theta) = a(1 - \cos(-\theta)) = g(\theta).$
Therefore, the curve is symmetrical about the y −axis.

Curve is not symmetrical about y- axis.
Curve is not symmetrical about the line $y = x$.
Curve is not symmetrical about the line $y = -x$.
Curve is not symmetrical in opposite quadrants.

2. Origin
$(0, 0)$: $x = f(\theta) = a(\theta - \sin\theta) = 0, y = g(\theta) = a(1 - \cos\theta) = 0$

$$\begin{cases} a(\theta - \sin\theta) = 0 \\ a(1-\cos\theta) = 0 \end{cases} => \begin{cases} \theta - \sin\theta = 0 \\ 1-\cos\theta = 0 \end{cases} => \begin{cases} \sin\theta = \theta \\ \cos\theta = 1 \end{cases} =>$$

$$=> \begin{cases} \sin\theta = \theta \\ \theta = 0 \quad \text{or} \quad \theta = 2\pi \end{cases} => \theta = 0$$

A curve passes through the origin.

Derivatives:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

At $\theta = 0, dy/dx = \infty$. Tangent to the curve at $\theta = 0$ is perpendicular to x –axis.

3. Intercepts

Intersection with x- axis: The points of intersection of the curve with the x – axis are given by the roots of $g(\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.

$a(1-\cos\theta) = 0$

$\cos\theta = 1$

$\theta = 0 \quad \text{or} \quad \theta = 2\pi$

$f(0) = a(0 - \sin(0)) = 0$

$f(2\pi) = a(2\pi - \sin 2\pi) = 2\pi a$

$Point(0,0), Point(2\pi, 0)$

$f(0) = a(0 - \sin(0)) = 0$

$f(2\pi) = a(2\pi - \sin 2\pi) = 2\pi a$

Then

$(x, y) = (0,0);\ (x, y) = (2a\pi, 0)$

Intersection with y- axis: The points of intersection of the curve with the y – axis are given by the roots of $f(\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.

$f(\theta) = 0 => a(\theta - \sin\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.

$\theta - \sin\theta = 0$

$\theta = 0$

$Point(0,0)$

$g(0) = 0$

Then

$(x, y) = (0,0)$

4. Asymptotes

$x = a(\theta - \sin\theta), y = a(1-\cos\theta)$

There is no vertical asymptote.

There is no horizontal asymptote.

There is no oblique asymptote.

5. Regions where no Part of the curve lies

$x = a(\theta - \sin\theta), y = a(1 - \cos\theta), a > 0$

Note that $y \geq 0$. Entire curve lies above the y −axis $(0 \leq y \leq 2a)$.

6. First derivative

$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

At $\theta = 0, dy/dx = \infty$. Tangent to the curve at $\theta = 0$ is perpendicular to x −axis.

At $\theta = \pi, dy/dx = 0$. Tangent to the curve is parallel to x −axis at $\theta = \pi$.

At $\theta = 2\pi, dy/dx = \infty$. Tangent to the curve is again perpendicular to x −axis at $\theta = 2\pi$.

For $0 < \theta < \pi, \frac{dy}{dx} > 0$.

Therefore, the function $y(x)$ is increasing in this interval.

For $\pi < \theta < 2\pi, \frac{dy}{dx} < 0$.

Therefore, the function $y(x)$ is decreasing in this interval.

7. Second derivative

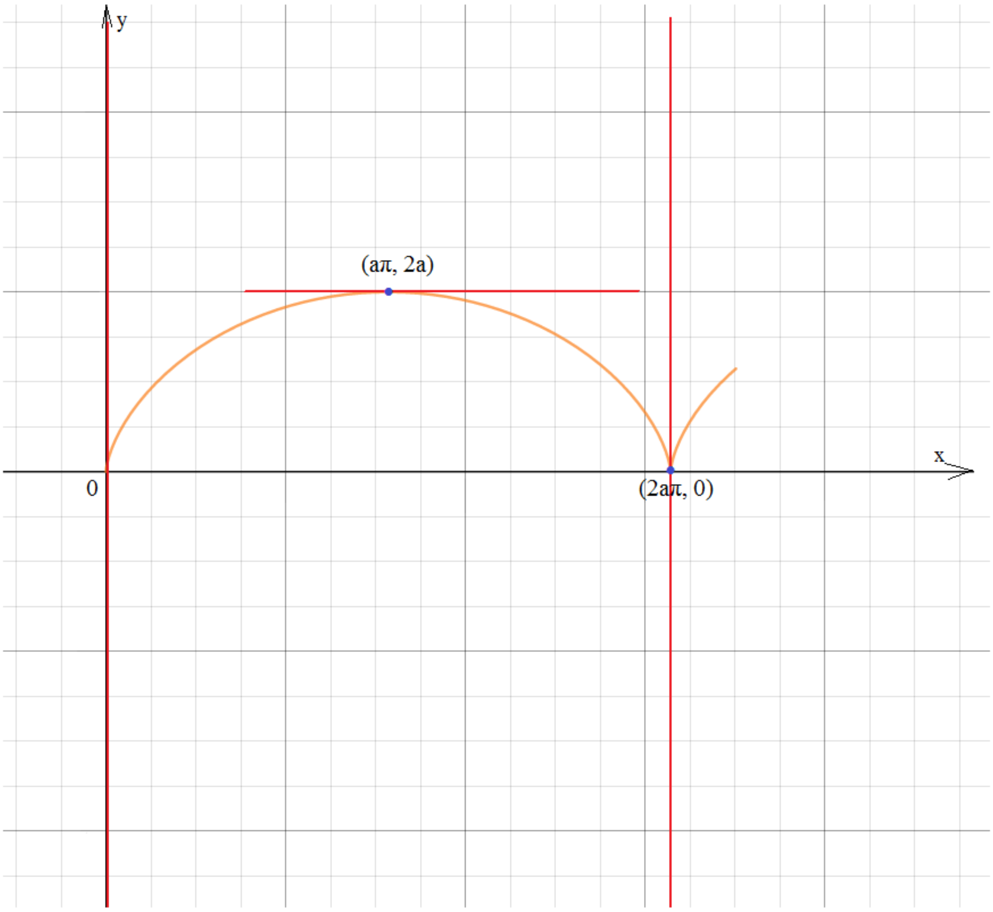
$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)}{\frac{dx}{d\theta}} = \frac{-\frac{1}{2\sin^2\frac{\theta}{2}}}{a(1 - \cos\theta)} = -\frac{1}{4\sin^2\frac{\theta}{2}}$$

For $0 < \theta < 2\pi, \frac{d^2y}{dx^2} < 0 => concave\ downward.$

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2 - 1)$	$a\pi$	$a(3\pi/2 + 1)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	∞


y
(aπ, 2a)
x
0
(2aπ, 0)