Answer on Question #74405 – Math – Calculus

Question

$$\iint_{y < x < \sqrt{4-y^2} \text{ and } 0 < y < \sqrt{2}} \frac{dx\,dy}{4+x^2+y^2} \text{ - ?}$$

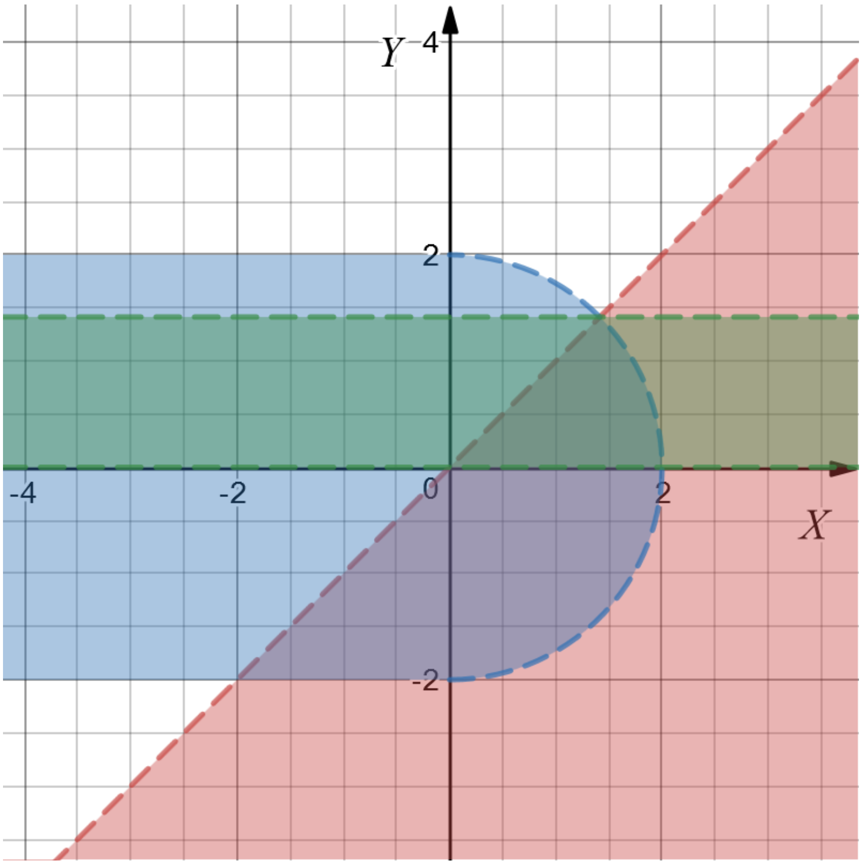
Solution

We depict the region of integration:

Red dotted line is $y = x$;

Blue dotted line is $x = \sqrt{4-y^2}$;

Green dotted lines are $y = 0, y = \sqrt{2}$;



$$\begin{cases} x = \sqrt{4-y^2} \\ y = x \\ y = \sqrt{2} \end{cases} \quad \begin{cases} x = \sqrt{2} \\ y = \sqrt{2} \end{cases};$$

We use the polar coordinates:

$$\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \end{cases}; r \in (0;2);\ \varphi \in \left(0;\frac{\pi}{4}\right); dx \cdot dy = rdr \cdot d\varphi$$

$$\iint\limits_{y<x<\sqrt{4-y^2}\ and\ 0<y<\sqrt{2}} \frac{dx\,dy}{4+x^2+y^2} =$$

$$= \int_0^{\frac{\pi}{4}} d\varphi \int_0^2 \frac{rdr}{4+(rcos(\varphi))^2+(rsin(\varphi))^2} =$$

$$= \int_0^{\frac{\pi}{4}} d\varphi \int_0^2 \frac{rdr}{4+r^2(cos^2\varphi+sin^2\varphi)} =$$

$$= \int_0^{\frac{\pi}{4}} d\varphi \int_0^2 \frac{rdr}{4+r^2} =$$

$$= \frac{\pi}{4}\int_0^2 \frac{rdr}{4+r^2} = \begin{bmatrix} t=4+r^2 \\ dt=2rdr \\ t\in(4;8) \end{bmatrix} = \frac{\pi}{4}\int_4^8 \frac{dt}{2t} =$$

$$= \frac{\pi}{4}\cdot\frac{1}{2}\cdot\int_4^8 \frac{dt}{t} = \frac{\pi}{8}(ln8-ln4) = \frac{\pi}{8}ln\frac{8}{4} = \frac{\pi}{8}ln2$$

Answer: $\iint\limits_{y<x<\sqrt{4-y^2}\ and\ 0<y<\sqrt{2}} \frac{dx\,dy}{4+x^2+y^2} = \frac{\pi}{8}\ln 2$