Answer on Question #74180 -Math - Calculus

Question

Give an example of a function which is not integrable on [0, pi].Justify your answer.

Solution

The example of a function which is not integrable on $[0;\pi]$ is $f(x) = 1/x$.

If the function $f(x)$ is integrable on $[a;b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Where

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

Let $f(x) = \frac{1}{x}, \quad a = 0, \quad b = \pi$, then $\Delta x = \frac{\pi}{n}$ and $x_i = i\frac{\pi}{n}$.

For given n we need to choose a set of points $x_i^* \in [x_{i-1}, x_i]$ that 'fails'. For example we can choose:

$$x_1^* = \frac{\pi}{n^2}, \quad x_i^* = x_i \quad (i \geq 2)$$

Then we have:

$$\sum_{i=1}^{n} f(x_i)\Delta x \geq f(x_1^*)\Delta x = \frac{1}{\pi/n^2} \cdot \frac{\pi}{n} = n$$

Here, we use the fact that every term in the sum is positive, so the sum is greater than

its first term $f(x_1^*)\Delta x$. If we let $n \to \infty$, the right-hand-side goes to ∞, and so by comparison we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \infty$$

The Riemann sum is divergent, so f is not integrable over $[0;\pi]$.

Answer: The function $f(x) = 1/x$ is not integrable on $[0;\pi]$.