ANSWER on Question #74169 – Math – Calculus

QUESTION

If

$$x \cdot \sin y = \sin(p + y),$$

show that

$$\sin p \cdot \frac{dy}{dx} + \sin^2 y = 0.$$

SOLUTION

We recall the rules for taking a derivative that we need to solve a given problem

$$1)\quad \frac{dy}{dx} \equiv y'$$

$$2)\quad f(x) = g(x) \to \frac{df}{dx} = \frac{dg}{dx}$$

$$3)\quad y = f(x) \cdot g(x) \to \frac{dy}{dx} = f'(x) \cdot g(x) + f(x) \cdot g'(x) - \mathit{the\ product\ rule}$$

$$4)\quad y = f\big(g(x)\big) \to \frac{dy}{dx} = f'\big(g(x)\big) \cdot g'(x) - \mathit{the\ chain\ rule}$$

$$5)\quad y = \sin x \to \frac{dy}{dx} = \cos x$$

$$6)\quad y = f(x + p) \to \frac{dy}{dx} = f'(x + p), \quad \forall p \in \mathbb{R}$$

$$7)\quad y = x \to \frac{dy}{dx} = (x)' = 1$$

(More information: https://en.wikipedia.org/wiki/Differentiation_rules)

In our case,

$$|x \cdot \sin y = \sin(p+y) \to \underbrace{\frac{d}{dx}(x \cdot \sin y)}_{the\ product\ rule} = \frac{d}{dx}(\sin(p+y)) \to$$

$$(x)' \cdot \sin y + x \cdot \frac{d(\sin y)}{dx} = \cos(p+y) \cdot \frac{dy}{dx} \to 1 \cdot \sin y + x \cdot \cos y \cdot \frac{dy}{dx} = \cos(p+y) \cdot \frac{dy}{dx}$$

$$\sin y + (x \cos y) \cdot \frac{dy}{dx} = \cos(p+y) \cdot \frac{dy}{dx} \to \sin y = \frac{dy}{dx} \cdot (\cos(p+y) - x \cos y)$$

According to the problem

$$x \cdot \sin y = \sin(p+y) \to \boxed{x = \frac{\sin(p+y)}{\sin y}}$$

Recall some formulas from trigonometry

$$1)\ \cos(p+y) = \cos p \cos y - \sin p \sin y$$

$$2)\ \ \sin(p+y) = \sin p \cos y + \cos p \sin y$$

$$3)\ \ \sin^2 \alpha + \cos^2 \alpha = 1,\ \ \forall \alpha \in \mathbb{R}$$

(More information: https://en.wikipedia.org/wiki/List_of_trigonometric_identities)

Then,

$$\sin y = \frac{dy}{dx} \cdot (\cos(y+p) - x \cos y) \to \sin y = \frac{dy}{dx} \cdot \left(\cos(p+y) - \frac{\sin(p+y)}{\sin y} \cdot \cos y\right) \to$$

$$\sin y = \frac{dy}{dx} \cdot \left(\frac{\cos(p+y) \cdot \sin y - \sin(p+y) \cdot \cos y}{\sin y}\right)$$

We transform the expression in parentheses

$$\frac{\cos(p+y)\cdot\sin y - \sin(p+y)\cdot\cos y}{\sin y} =$$

$$= \frac{(\cos p\cos y - \sin p\sin y)\cdot\sin y - (\sin p\cos y + \cos p\sin y)\cdot\cos y}{\sin y} =$$

$$= \frac{\cos p\cos y\sin y - \sin p\sin^2 y - \sin p\cos^2 y - \cos p\cos y\sin y}{\sin y} =$$

$$= \frac{-\sin p\cdot\overbrace{(\sin^2 y + \cos^2 y)}^{=1}}{\sin y} = \frac{-\sin p\cdot 1}{\sin y} = \frac{-\sin p}{\sin y}$$

By collecting all the formulas obtained in one, we obtain

$$\sin y = \frac{dy}{dx}\cdot\left(\frac{\cos(p+y)\cdot\sin y - \sin(p+y)\cdot\cos y}{\sin y}\right) \to \sin y = \frac{dy}{dx}\cdot\left(\frac{-\sin p}{\sin y}\right) \to$$

$$\sin y = \frac{dy}{dx}\cdot\left(\frac{-\sin p}{\sin y}\right)\Bigg| \times (\sin y) \to \sin^2 y = \sin y\cdot\frac{-\sin p}{\sin y}\cdot\frac{dy}{dx} \to \sin^2 y = -\sin p\cdot\frac{dy}{dx} \to$$

$$\boxed{\sin^2 y + \sin p\cdot\frac{dy}{dx} = 0 \leftrightarrow \sin p\cdot\frac{dy}{dx} + \sin^2 y = 0}$$

Q.E.D.