Answer on Question #74167, Math / Calculus

Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$. State the properties you use for tracing it also.

Solution

We use the following properties for tracing the curve:

We have the Cartesian curve defined by the parametric equations $x = f(\theta)$, $y = g(\theta)$. Since y is a periodic function of θ with period 2π, it is sufficient to trace the curve for $\theta \in [0, 2\pi]$.

For $\theta \in [0, 2\pi], x$ and y are well defined.

Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta); 0 \le \theta \le 2\pi; a > 0$.

1. Symmetry

$x = f(\theta) = a(\theta - \sin\theta);$

$y = g(\theta) = a(1 - \cos\theta)$

$f(-\theta) = a(-\theta - \sin(-\theta)) = -f(\theta);$

$g(-\theta) = a(1 - \cos(-\theta)) = g(\theta).$

Therefore, the curve is symmetrical about the y −axis.

Curve is not symmetrical about y- axis.

Curve is not symmetrical about the line $y = x$.

Curve is not symmetrical about the line $y = -x$.

Curve is not symmetrical in opposite quadrants.

2. Origin

$(0,0)$: $x = f(\theta) = a(\theta - \sin\theta) = 0, y = g(\theta) = a(1 - \cos\theta) = 0$

$$\begin{cases} a(\theta - \sin\theta) = 0 \\ a(1 - \cos\theta) = 0 \end{cases} => \begin{cases} \theta - \sin\theta = 0 \\ 1 - \cos\theta = 0 \end{cases} => \begin{cases} \sin\theta = \theta \\ \cos\theta = 1 \end{cases} =>$$

$$=> \begin{cases} \sin\theta = \theta \\ \theta = 0 \quad \text{or} \quad \theta = 2\pi \end{cases} => \theta = 0$$

A curve passes through the origin.

Derivatives:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

At $\theta = 0, dy/dx = \infty$. Tangent to the curve at $\theta = 0$ is perpendicular to x −axis.

3. Intercepts

Intersection with x- axis: The points of intersection of the curve with the x – axis are given by the roots of $g(\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.

$a(1 - \cos\theta) = 0$

$\cos\theta = 1$

$\theta = 0 \quad \text{or} \quad \theta = 2\pi$

$f(0) = a(0 - \sin(0)) = 0$
$f(2\pi) = a(2\pi - \sin 2\pi) = 2\pi a$
$Point(0, 0), Point(2\pi, 0)$

Intersection with y- axis: The points of intersection of the curve with the y – axis are given by the roots of $f(\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.
$f(\theta) = 0 => a(\theta - \sin\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.
$\theta - \sin\theta = 0$
$\theta = 0$
$Point(0, 0)$
$g(\theta) = 0 => a(1 - \cos\theta) = 0, 0 \le \theta \le 2\pi; a > 0$.
$\theta = 0, \quad \theta = 2\pi$
$f(0) = a(0 - \sin(0)) = 0$
$f(2\pi) = a(2\pi - \sin 2\pi) = 2\pi a$

4. Asymptotes
$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$
There is no vertical asymptote.
There is no horizontal asymptote.
There is no oblique asymptote.

5. Regions where no Part of the curve lies
Note that $y \ge 0$. Entire curve lies above the y −axis $(0 \le y \le 2a)$.

6. First derivative
$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

At $\theta = 0, dy/dx = \infty$. Tangent to the curve at $\theta = 0$ is perpendicular to x −axis.

At $\theta = \pi, dy/dx = 0$. Tangent to the curve is parallel to x −axis at $\theta = \pi$.

At $\theta = 2\pi, dy/dx = \infty$. Tangent to the curve is again perpendicular to x −axis at $\theta = 2\pi$.

For $0 < \theta < \pi, \frac{dy}{dx} > 0$.
Therefore, the function $y(x)$ is increasing in this interval.

For $\pi < \theta < 2\pi, \frac{dy}{dx} < 0$.
Therefore, the function $y(x)$ is decreasing in this interval.

7. Second derivative

$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \cot\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)}{\frac{dx}{d\theta}} = \frac{-\frac{1}{2\sin^2\frac{\theta}{2}}}{a(1-\cos\theta)} = -\frac{1}{4\sin^2\frac{\theta}{2}}$$

For $0 < \theta < 2\pi, \frac{d^2y}{dx^2} < 0 => concave\ downward.$

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2-1)$	$a\pi$	$a(3\pi/2+1)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	∞