Answer on Question #73992 – Math – Calculus

Question

Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$. State the properties you use for tracing it, also.

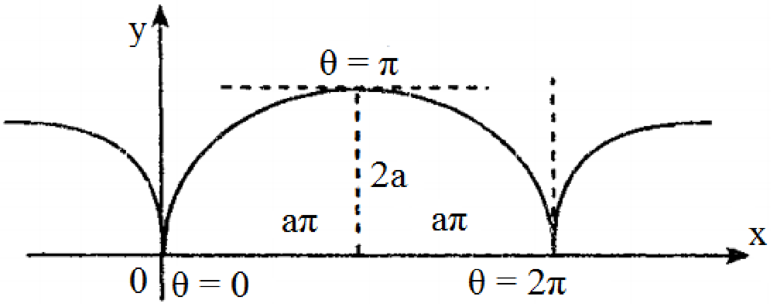
Solution

To trace a Cartesian curve defined by the parametric equations $x = f(\theta)$, $y = g(\theta)$, we use the following properties.
If either x or y is a periodic function of θ with period T then trace the curve in one period, say for $\theta \in [0, T]$.
Trace the curve $x = a(\theta - \sin\theta), y = a(1 - \cos\theta); 0 \leq \theta \leq 2\pi; a > 0$.
Note that
$f(-\theta) = a(-\theta - \sin(-\theta)) = -f(\theta)$;
$g(-\theta) = a(1 - \cos(-\theta)) = g(\theta)$.
Therefore, the curve is symmetric about the y −axis.
Also y is a periodic function of θ with period 2π. It is sufficient to trace the curve for $\theta \in [0, 2\pi]$.
For $\theta \in [0, 2\pi], x$ and y are well defined.
Note that $y \geq 0$. Entire curve lies above the y −axis $(0 \leq y \leq 2a)$.

Determine the points where the curve crosses the axes. The points of intersection of the curve with the x – axis are given by the roots of $f(\theta) = 0$, while those with the y – axis are given by the roots of $g(\theta) = 0$.
$f(\theta) = 0 => a(\theta - \sin\theta) = 0, 0 \leq \theta \leq 2\pi; a > 0$.
$\theta = 0 \text{ or } \theta = \sin\theta \ , 0 < \theta < \frac{\pi}{2}$
$g(\theta) = 0 => a(1 - \cos\theta) = 0, 0 \leq \theta \leq 2\pi; a > 0$.
$\theta = 0, \ \ \theta = 2\pi$
$f(0) = a(0 - \sin(0)) = 0$
$f(2\pi) = a(2\pi - \sin 2\pi) = 2\pi a$
Derivatives:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2 - 1)$	$a\pi$	$a(3\pi/2 + 1)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	∞



At $\theta = 0, dy/dx = \infty$. Tangent to the curve at $\theta = 0$ is perpendicular to x −axis.

At $\theta = \pi, dy/dx = 0$. Tangent to the curve is parallel to x −axis at $\theta = \pi$.

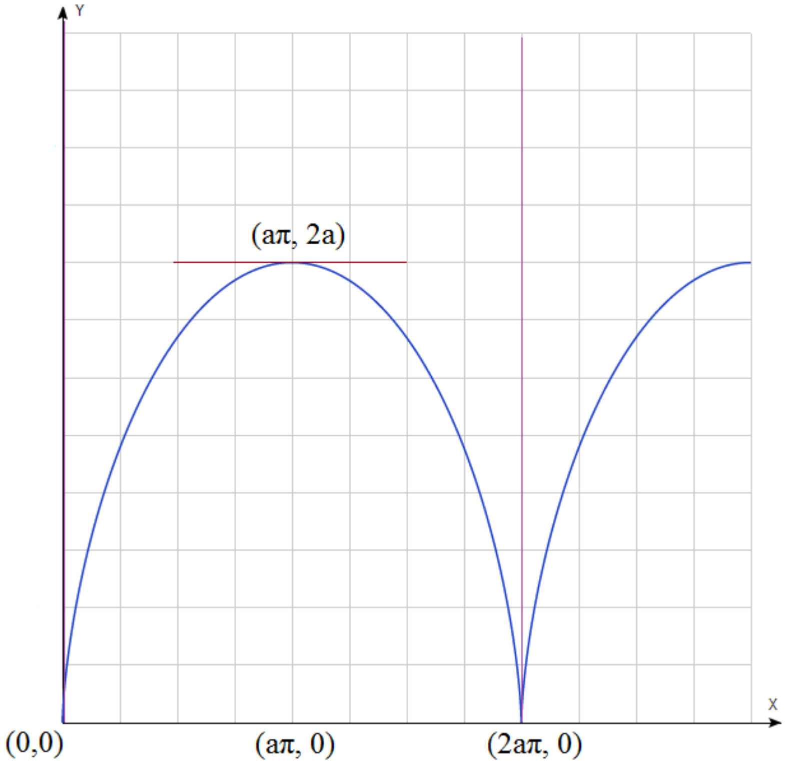
At $\theta = 2\pi, dy/dx = \infty$. Tangent to the curve is again perpendicular to x −axis at $\theta = 2\pi$.

For $0 < \theta < \pi, \dfrac{dy}{dx} > 0$.

Therefore, the function is increasing in this interval.

For $\pi < \theta < 2\pi, \dfrac{dy}{dx} < 0$.

Therefore, the function is decreasing in this interval.



$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)}{\frac{dx}{d\theta}} = \frac{-\frac{1}{2\sin^2\frac{\theta}{2}}}{a(1-\cos\theta)} = -\frac{1}{4\sin^2\frac{\theta}{2}}$$

For $0 < \theta < 2\pi, \dfrac{d^2y}{dx^2} < 0 \Rightarrow concave\ downward.$

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