Answer on Question #73030 – Math – Calculus

Question

Solve the following differential equation

$$(D-1)^2(D^2+1)^2y = sin^2\frac{x}{2} + e^x + x$$

Solution

Substitute

$$y = e^{kx}$$

Then

$$Dy = \frac{dy}{dx} = ke^{kx}$$

$$D^2y = \frac{d^2y}{dx^2} = k^2e^{kx}$$

$$D^4y = \frac{d^4y}{dx^4} = k^4e^{kx}$$

$$(y'' - 2y' + y)\left(y^{(4)} + 2y'' + y\right) = 0$$

$$(k^2e^{kx} - 2ke^{kx} + e^{kx})(k^4e^{kx} + 2k^2e^{kx} + e^{kx}) = 0$$

$$e^{kx}(k^2 - 2k + 1)(k^4 + 2k^2 + 1) = 0$$

$$(k-1)^2(k^2+1)^2 = 0$$

$$k = 1; k = i; k = -i$$

So, we have three fundamental solutions:

$$e^x;\ e^{ix};\ e^{-ix}$$

Each of those three fundamental solutions satisfies the homogeneous equation and also any linear combination of those.

Each of roots k is double and to obtain three more fundamental solutions we need to multiply the corresponding fundamental solution by x :

$$xe^x; x\sin x; x\cos x$$

So the general solution to the homogeneous equation is

$$Y = Ae^x + Bxe^x + C\sin x + Dx\sin x + E\cos x + Fx\cos x$$

where A, B, C, D, E, F are arbitrary constants.

Since

$$sin^2\frac{x}{2} = \frac{1}{2} - \frac{\cos x}{2}$$

$$(y'' - 2y' + y)\left(y^{(4)} + 2y'' + y\right) = \frac{1}{2} - \frac{\cos x}{2} + e^x + x$$

To get e^x a particular solution

$$\tilde{y}_1 = ax^2e^x$$

$$\tilde{y}_1{}' = 2axe^x + ax^2e^x = ae^x(2x + x^2)$$

$$\tilde{y}_1{}'' = ae^x(2x + x^2) + ae^x(2 + 2x) = ae^x(x^2 + 4x + 2)$$

$$\tilde{y}_1{}^{(3)} = ae^x(x^2 + 4x + 2) + ae^x(2x + 4) = ae^x(x^2 + 6x + 6)$$

$$\tilde{y}_1{}^{(4)} = ae^x(x^2 + 6x + 6) + ae^x(2x + 6) = ae^x(x^2 + 8x + 12)$$

$$(ae^x(x^2 + 4x + 2) - 2ae^x(2x + x^2) + ax^2e^x) \times$$

$$\times\,(ae^x(x^2 + 8x + 12) + 2ae^x(x^2 + 4x + 2) + ax^2e^x) = e^x$$

$$a(2 \cdot 12 + 2 \cdot 4) = 1$$

$$a = \frac{1}{32}$$

$$\tilde{y}_1 = \frac{x^2e^x}{32}$$

To get $\left(-\frac{\cos x}{2}\right)$ a particular solution

$$\tilde{y}_2 = bx^2\cos x$$

$$\tilde{y}_2' = 2bx\cos x - bx^2 \sin x$$

$$\tilde{y}_2'' = 2b\cos x - 4bx\sin x - bx^2\cos x = b\cos x\,(2 - x^2) - 4bx\sin x$$

$$\tilde{y}_2{}^{(3)} = -2bx\cos x - b\sin x(2 - x^2) - 4b\sin x - 4bx\cos x = -6bx\cos x - b\sin x\,(6 - x^2)$$

$$\tilde{y}_2{}^{(4)} = -6b\cos x + 6bx\sin x + 2bx\sin x - b(6 - x^2)\cos x = b(x^2 - 12)\cos x + 8bx\sin x$$

$$(b\cos x\,(2 - x^2) - 4bx\sin x - 2(2bx\cos x - bx^2\sin x) + bx^2\cos x) \times$$

$$\times\,(b(x^2 - 12)\cos x + 8bx\sin x + 2(b\cos x\,(2 - x^2) - 4bx\sin x) + bx^2\cos x) = -\frac{\cos x}{2}$$

$$(-2bx\cos x - 4bx\sin x + 2bx^2\sin x)(-8b\cos x) = -\frac{\cos x}{2}$$

$$b = 0$$

$$\tilde{y}_2 = 0$$

To get $\left(x + \frac{1}{2}\right)$ a particular solution

$$\tilde{y}_3 = cx + d$$

$$\tilde{y}_3' = c$$

$$\tilde{y}_3'' = \tilde{y}_3{}^{(4)} = 0$$

$$(-2c + cx + d)(cx + d) = x + \frac{1}{2}$$

$$-2c^2 + 2cd = 1$$

$$-2cd + d^2 = \frac{1}{2}$$

$$c^2 = 0$$

$$\tilde{y}_3 = 0$$

$$\tilde{y} = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 = \frac{x^2 e^x}{32}$$

Answer:

$$y = Y + \tilde{y} = Ae^x + Bxe^x + C\sin x + Dx\sin x + E\cos x + Fx\cos x + \frac{x^2 e^x}{32}$$

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