Answer on Question #72164 – Math – Calculus

1. Compute the divergence and curl of each of the following vector fields:

a)

$$F = (x^2 + y^2, x^2 - y^2, z^2)$$

Solution

$$div\ F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 2x - 2y + 2z$$

$$curl\ F = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = (2x + 2y)\bar{k}$$

b)

$$F = (x + y, x - y, z)$$

Solution

$$div\ F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 - 1 + 1 = 1$$

$$curl\ F = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

2. For any two vector fields F, G show that:

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

Solution

$$F \times G = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} = \left(F_y \cdot G_z - F_z \cdot G_y\right)\bar{\imath} - (F_x \cdot G_z - F_z \cdot G_x)\bar{\jmath} + \left(F_x \cdot G_y - F_y \cdot G_x\right)\bar{k}$$

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$$\nabla\cdot(F\times G)=\frac{\partial\left(F_y\cdot G_z-F_z\cdot G_y\right)}{\partial x}-\frac{\partial(F_x\cdot G_z-F_z\cdot G_x)}{\partial y}+\frac{\partial\left(F_x\cdot G_y-F_y\cdot G_x\right)}{\partial z}=$$

$$=G_z\cdot\frac{\partial F_y}{\partial x}+F_y\cdot\frac{\partial G_z}{\partial x}-G_y\cdot\frac{\partial F_z}{\partial x}-F_z\cdot\frac{G_y}{\partial x}-G_z\cdot\frac{\partial F_x}{\partial y}-F_x\cdot\frac{\partial G_z}{\partial y}+G_x\cdot\frac{\partial F_z}{\partial y}+F_z\cdot\frac{\partial G_x}{\partial y}+$$

$$+G_y\cdot\frac{\partial F_x}{\partial z}+F_x\cdot\frac{\partial G_y}{\partial z}-G_x\cdot\frac{\partial F_y}{\partial z}-F_y\cdot\frac{\partial G_x}{\partial z}=$$

$$=G_x\cdot\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)-G_y\cdot\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)+G_z\cdot\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)-F_x\cdot\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}\right)+$$
$$+F_y\cdot\left(\frac{\partial G_z}{\partial x}-\frac{\partial G_x}{\partial z}\right)-F_z\cdot\left(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right)$$

$$\nabla\times F=\begin{vmatrix}\bar\imath & \bar\jmath & \bar k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\bar\imath-\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\bar\jmath+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\bar k$$

$$G\cdot(\nabla\times F)=G_x\cdot\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)-G_y\cdot\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)+G_z\cdot\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

Analogically,

$$F\cdot(\nabla\times G)=F_x\cdot\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}\right)-F_y\cdot\left(\frac{\partial G_z}{\partial x}-\frac{\partial G_x}{\partial z}\right)+F_z\cdot\left(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right)$$

Finally,

$$G\cdot(\nabla\times F)-F\cdot(\nabla\times G)=G_x\cdot\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)-G_y\cdot\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)+G_z\cdot\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)-$$

$$-F_x\cdot\left(\frac{\partial G_z}{\partial y}-\frac{\partial G_y}{\partial z}\right)+F_y\cdot\left(\frac{\partial G_z}{\partial x}-\frac{\partial G_x}{\partial z}\right)-F_z\cdot\left(\frac{\partial G_y}{\partial x}-\frac{\partial G_x}{\partial y}\right)=\nabla\cdot(F\times G)$$

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3. Given scalar functions $u(x,y,z), v(x,y,z), w(x,y,z), \varphi(x,y,z)$ and a vector field

$$F(x,y,z) \; = u(x,y,z), v(x,y,z), w(x,y,z),$$

prove that

$$div(\varphi F) = \varphi(div\ F) + (\nabla\varphi)\cdot F$$

Solution

$$div\ F = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\varphi(div\ F) = \varphi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\nabla\varphi = \frac{\partial \varphi}{\partial x}\bar{\imath} + \frac{\partial \varphi}{\partial y}\bar{\jmath} + \frac{\partial \varphi}{\partial z}\bar{k}$$

$$(\nabla\varphi)\cdot F = \frac{\partial \varphi}{\partial x}\cdot u + \frac{\partial \varphi}{\partial y}\cdot v + \frac{\partial \varphi}{\partial z}\cdot w$$

$$\varphi(div\ F) + (\nabla\varphi)\cdot F = \varphi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial \varphi}{\partial x}\cdot u + \frac{\partial \varphi}{\partial y}\cdot v + \frac{\partial \varphi}{\partial z}\cdot w$$

$$div(\varphi F) = div(\varphi u + \varphi v + \varphi w) = \frac{\partial(\varphi u)}{\partial x} + \frac{\partial(\varphi v)}{\partial y} + \frac{\partial(\varphi w)}{\partial z} =$$

$$= \frac{\partial \varphi}{\partial x}\cdot u + \frac{\partial \varphi}{\partial y}\cdot v + \frac{\partial \varphi}{\partial z}\cdot w + \frac{\partial u}{\partial x}\cdot\varphi + \frac{\partial v}{\partial y}\cdot\varphi + \frac{\partial w}{\partial z}\cdot\varphi =$$

$$= \varphi(div\ F) + (\nabla\varphi)\cdot F$$

4. Find a function f such that $F = \nabla f$, where $F = (yz, xz, xy + 2z)$

Solution

$$\nabla f = \frac{\partial f}{\partial x}\bar{\imath} + \frac{\partial f}{\partial y}\bar{\jmath} + \frac{\partial f}{\partial z}\bar{k} = F$$

$$\frac{\partial f}{\partial x} = yz\ ;\ \frac{\partial f}{\partial y} = xz\ ;\ \frac{\partial f}{\partial z} = xy + 2z$$

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Answer:

$$f = (xyz, xyz, xyz + z^2)$$

5. Let $r = (x, y, z)$ and $r = |r|$, verify that

a)

$$\nabla \cdot r = 3$$

Solution

$$\nabla \cdot r = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = 1 + 1 + 1 = 3$$

b)

$$div(r^n \cdot r) = (n+3)r^n$$

Solution

$$r^n \cdot r = \left(\sqrt{x^2+y^2+z^2}\right)^n \cdot (x,y,z) = \left[x\left(\sqrt{x^2+y^2+z^2}\right)^n, y\left(\sqrt{x^2+y^2+z^2}\right)^n, z\left(\sqrt{x^2+y^2+z^2}\right)^n\right]$$

$$div(r^n \cdot r) = \left(\sqrt{x^2+y^2+z^2}\right)^n + x \cdot 2x \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} +$$

$$+\left(\sqrt{x^2+y^2+z^2}\right)^n + y \cdot 2y \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} +$$

$$+\left(\sqrt{x^2+y^2+z^2}\right)^n + z \cdot 2z \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} =$$

$$= 3r^n + nr^{2\left(\frac{n}{2}-1\right)}(x^2+y^2+z^2) = 3r^n + nr^{n-2}r^2 = 3r^n + nr^n = (n+3)r^n$$

6. Given $u = xy^2z^2$ and $v = yz - 3x^2$, find $\nabla \cdot [(\nabla u) \times (\nabla v)]$

Solution

$$\nabla u = \frac{\partial u}{\partial x}\bar{\imath} + \frac{\partial u}{\partial y}\bar{\jmath} + \frac{\partial u}{\partial z}\bar{k} = (y^2z^2)\bar{\imath} + (2xyz^2)\bar{\jmath} + (2xy^2z)\bar{k}$$

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$$\nabla v = \frac{\partial v}{\partial x}\bar{\imath} + \frac{\partial v}{\partial y}\bar{\jmath} + \frac{\partial v}{\partial z}\bar{k} = (-6x)\bar{\imath} + (z)\bar{\jmath} + (y)\bar{k}$$

$$(\nabla u) \times (\nabla v) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ (\nabla u)_x & (\nabla u)_y & (\nabla u)_z \\ (\nabla v)_x & (\nabla v)_y & (\nabla v)_z \end{vmatrix} = (2xyz^2 \cdot y - z \cdot 2xy^2 z)\bar{\imath} - (y^2z^2 \cdot y + 2xy^2 z \cdot 6x)\bar{\jmath} +$$

$$+(y^2z^2 \cdot z + 2xyz^2 \cdot 6x)\bar{k} = -(y^3z^2 + 12x^2y^2z)\bar{\jmath} + (y^2z^3 + 12x^2yz^2)\bar{k}$$

$$\nabla \cdot [(\nabla u) \times (\nabla v)] = \frac{\partial[(\nabla u) \times (\nabla v)]_x}{\partial x} + \frac{\partial[(\nabla u) \times (\nabla v)]_y}{\partial y} + \frac{\partial[(\nabla u) \times (\nabla v)]_z}{\partial z}$$

Answer:

$$\nabla \cdot [(\nabla u) \times (\nabla v)] = -3y^2z^2 - 24x^2yz + 3y^2z^2 + 24x^2yz = 0$$

7. Given the vector field $F = (e^x \sin y, e^x \cos y, z)$, show that F is irrotational, hence or otherwise, find the functions f such that $\nabla f = F$

Solution

$$curl\, F = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & z \end{vmatrix} =$$

$$= (e^x \cos y - e^x \cos y)\bar{k} = 0$$

$$\nabla f = \frac{\partial f}{\partial x}\bar{\imath} + \frac{\partial f}{\partial y}\bar{\jmath} + \frac{\partial f}{\partial z}\bar{k} = F$$

$$\frac{\partial f}{\partial x} = e^x \sin y \ ; \ \frac{\partial f}{\partial y} = e^x \cos y \ ; \ \frac{\partial f}{\partial z} = z$$

Answer:

$$f = \left(e^x \sin y, e^x \sin y, \frac{z^2}{2}\right)$$