ANSWER on Question #71592 – Math – Analytic Geometry

QUESTION

1) Find the coordinates of the centroid (one type of "centre of a triangle")
2) Find the coordinates of the orthocentre (a second type of "centre of a triangle")

3) Find the coordinates of the circumcentre (a third type of "centre of a triangle")

State any relationship you see between the three key points you found above.

SOLUTION

Suppose we are given a triangle ΔABC with coordinates of the vertices:

$$\begin{cases} A(A_x; A_y) \\ B(B_x; B_y) \\ C(C_x; C_y) \end{cases}$$

1) Find the coordinates of the centroid $M(M_x; M_y)$

1 step: We find the coordinates of $N(N_x; N_y)$ - the middle of the side BC.

$$BN = CN \rightarrow \begin{cases} N_x = \dfrac{B_x + C_x}{2} \\ N_y = \dfrac{B_y + C_y}{2} \end{cases}$$

(More information: https://en.wikipedia.org/wiki/Midpoint)

2 step: Let us find the coordinates of the $M(M_x; M_y)$ - centroid of the triangle ΔABC.

The main property of the triangle's centroid: The centroid is twice as close along any median to the side that the median intersects as it is to the vertex it emanates from.

(More information: https://en.wikipedia.org/wiki/Median_(geometry))

Then,

$$AM:MN = 2:1 \rightarrow \begin{cases} M_x = \dfrac{2 \cdot N_x + 1 \cdot A_x}{2+1} \\ M_y = \dfrac{2 \cdot N_y + 1 \cdot A_y}{2+1} \end{cases} \rightarrow \begin{cases} M_x = \dfrac{2 \cdot \dfrac{B_x + C_x}{2} + 1 \cdot A_x}{2+1} \\ M_y = \dfrac{2 \cdot \dfrac{B_y + C_y}{2} + 1 \cdot A_y}{2+1} \end{cases} \rightarrow$$

$$\begin{cases} M_x = \dfrac{A_x + B_x + C_x}{3} \\ M_y = \dfrac{A_y + B_y + C_y}{3} \end{cases}$$

Conclusion,

$$\boxed{\begin{cases} A(A_x; A_y) \\ B(B_x; B_y) \\ C(C_x; C_y) \\ M(M_x; M_y) - centroid\ of\ the\ triangle\ \Delta ABC \end{cases} \rightarrow \begin{cases} M_x = \dfrac{A_x + B_x + C_x}{3} \\ M_y = \dfrac{A_y + B_y + C_y}{3} \end{cases}}$$

3) Find the coordinates of the circumcenter $U(U_x;U_y)$

By the definition, the circumcenter $U(U_x;U_y)$ - point is equidistant from all its vertices. We write the equation of the circumscribed of a circle

$$(X-U_x)^2+(Y-U_y)^2=R^2$$

(More information: https://en.wikipedia.org/wiki/Circle)

Since the vertex of a triangle is on a given circle, their coordinates satisfy this equation:

$$\begin{cases}(A_x-U_x)^2+(A_y-U_y)^2=R^2\\(B_x-U_x)^2+(B_y-U_y)^2=R^2\\(C_x-U_x)^2+(C_y-U_y)^2=R^2\end{cases}$$

We have obtained a system of three equations with respect to three unknowns: $U_x;U_y;R$.

We will try to solve this system and find the coordinates of the center.

$$\begin{cases}(A_x-U_x)^2+(A_y-U_y)^2=R^2\\(B_x-U_x)^2+(B_y-U_y)^2=R^2\\(C_x-U_x)^2+(C_y-U_y)^2=R^2\end{cases}\rightarrow$$

$$\begin{cases}(A_x-U_x)^2+(A_y-U_y)^2=(B_x-U_x)^2+(B_y-U_y)^2\\(A_x-U_x)^2+(A_y-U_y)^2=(C_x-U_x)^2+(C_y-U_y)^2\end{cases}\rightarrow$$

$$\begin{cases}A_x^2-2A_xU_x+U_x^2+A_y^2-2A_yU_y+U_y^2=B_x^2-2B_xU_x+U_x^2+B_y^2-2B_yU_y+U_y^2\\A_x^2-2A_xU_x+U_x^2+A_y^2-2A_yU_y+U_y^2=C_x^2-2C_xU_x+U_x^2+C_y^2-2C_yU_y+U_y^2\end{cases}$$

$$\begin{cases}-2A_xU_x+2B_xU_x-2A_yU_y+2B_yU_y=B_x^2-A_x^2+B_y^2-A_y^2\\-2A_xU_x+2C_xU_x-2A_yU_y+2C_yU_y=C_x^2-A_x^2+C_y^2-A_y^2\end{cases}$$

$$\begin{cases}2U_x(B_x-A_x)+2U_y(B_y-A_y)=B_x^2-A_x^2+B_y^2-A_y^2\\2U_x(C_x-A_x)+2U_y(C_y-A_y)=C_x^2-A_x^2+C_y^2-A_y^2\end{cases}$$

1 step: From this system we find U_x

$$\begin{cases}2U_x(B_x-A_x)+2U_y(B_y-A_y)=B_x^2-A_x^2+B_y^2-A_y^2 \,\Big|\times(C_y-A_y)\\ 2U_x(C_x-A_x)+2U_y(C_y-A_y)=C_x^2-A_x^2+C_y^2-A_y^2 \,\Big|\times(B_y-A_y)\end{cases}$$

$$\begin{cases}2U_x(B_x-A_x)(C_y-A_y)+2U_y(B_y-A_y)(C_y-A_y)=(B_x^2-A_x^2+B_y^2-A_y^2)(C_y-A_y)\\ 2U_x(C_x-A_x)(B_y-A_y)+2U_y(C_y-A_y)(B_y-A_y)=(C_x^2-A_x^2+C_y^2-A_y^2)(B_y-A_y)\end{cases}$$

$$2U_x(B_x-A_x)(C_y-A_y)-2U_x(C_x-A_x)(B_y-A_y)=$$

$$=(B_x^2-A_x^2+B_y^2-A_y^2)(C_y-A_y)-(C_x^2-A_x^2+C_y^2-A_y^2)(B_y-A_y)\rightarrow$$

$$U_x=\frac{(B_x^2-A_x^2+B_y^2-A_y^2)(C_y-A_y)-(C_x^2-A_x^2+C_y^2-A_y^2)(B_y-A_y)}{2\Big((B_x-A_x)(C_y-A_y)-(C_x-A_x)(B_y-A_y)\Big)}$$

We transform this expression a little:

$$(B_x-A_x)(C_y-A_y)-(C_x-A_x)(B_y-A_y)=$$

$$=B_xC_y-B_xA_y-A_xC_y+A_xA_y-C_xB_y+C_xA_y+A_xB_y-A_xA_y=$$

$$=(A_xB_y-A_xC_y)+(B_xC_y-B_xA_y)+(C_xA_y-C_xB_y)=$$

$$=A_x(B_y-C_y)+B_x(C_y-A_y)+C_x(A_y-B_y)$$

$$\boxed{(B_x-A_x)(C_y-A_y)-(C_x-A_x)(B_y-A_y)=A_x(B_y-C_y)+B_x(C_y-A_y)+C_x(A_y-B_y)}$$

$$(B_x^2-A_x^2+B_y^2-A_y^2)(C_y-A_y)-(C_x^2-A_x^2+C_y^2-A_y^2)(B_y-A_y)=$$

$$=(B_x^2+B_y^2)(C_y-A_y)-(A_x^2+A_y^2)(C_y-A_y)-$$

$$-(C_x^2+C_y^2)(B_y-A_y)+(A_x^2+A_y^2)(B_y-A_y)=$$

$$=(A_x^2+A_y^2)(B_y-A_y-C_y+A_y)+(B_x^2+B_y^2)(C_y-A_y)+(C_x^2+C_y^2)(A_y-B_y)=$$

$$=(A_x^2+A_y^2)(B_y-C_y)+(B_x^2+B_y^2)(C_y-A_y)+(C_x^2+C_y^2)(A_y-B_y)$$

$$\boxed{\left(B_x^2 - A_x^2 + B_y^2 - A_y^2\right)\left(C_y - A_y\right) - \left(C_x^2 - A_x^2 + C_y^2 - A_y^2\right)\left(B_y - A_y\right) =}$$

$$\boxed{= \left(A_x^2 + A_y^2\right)\left(B_y - C_y\right) + \left(B_x^2 + B_y^2\right)\left(C_y - A_y\right) + \left(C_x^2 + C_y^2\right)\left(A_y - B_y\right)}$$

Then,

$$\boxed{U_x = \frac{\left(A_x^2 + A_y^2\right)\left(B_y - C_y\right) + \left(B_x^2 + B_y^2\right)\left(C_y - A_y\right) + \left(C_x^2 + C_y^2\right)\left(A_y - B_y\right)}{2\left(A_x\left(B_y - C_y\right) + B_x\left(C_y - A_y\right) + C_x\left(A_y - B_y\right)\right)}}$$

2 step: From this system we find U_y

$$\begin{cases} 2U_x(B_x - A_x) + 2U_y\left(B_y - A_y\right) = B_x^2 - A_x^2 + B_y^2 - A_y^2 \middle| \times (C_x - A_x) \\ 2U_x(C_x - A_x) + 2U_y\left(C_y - A_y\right) = C_x^2 - A_x^2 + C_y^2 - A_y^2 \middle| \times (B_x - A_x) \end{cases}$$

$$\begin{cases} 2U_x(B_x - A_x)(C_x - A_x) + 2U_y\left(B_y - A_y\right)(C_x - A_x) = \left(B_x^2 - A_x^2 + B_y^2 - A_y^2\right)(C_x - A_x) \\ 2U_x(C_x - A_x)(B_x - A_x) + 2U_y\left(C_y - A_y\right)(B_x - A_x) = \left(C_x^2 - A_x^2 + C_y^2 - A_y^2\right)(B_x - A_x) \end{cases}$$

$$2U_y\left(B_y - A_y\right)(C_x - A_x) - 2U_x\left(C_y - A_y\right)(B_x - A_x) =$$

$$= \left(B_x^2 - A_x^2 + B_y^2 - A_y^2\right)(C_x - A_x) - \left(C_x^2 - A_x^2 + C_y^2 - A_y^2\right)(B_x - A_x) \rightarrow$$

$$U_y = \frac{\left(B_x^2 - A_x^2 + B_y^2 - A_y^2\right)(C_x - A_x) - \left(C_x^2 - A_x^2 + C_y^2 - A_y^2\right)(B_x - A_x)}{2\left(\left(B_y - A_y\right)(C_x - A_x) - \left(C_y - A_y\right)(B_x - A_x)\right)}$$

Applying similar transformations, we obtain

$$\boxed{U_y = \frac{\left(A_x^2 + A_y^2\right)(C_x - B_x) + \left(B_x^2 + B_y^2\right)(A_x - C_x) + \left(C_x^2 + C_y^2\right)(B_x - A_x)}{2\left(A_x\left(B_y - C_y\right) + B_x\left(C_y - A_y\right) + C_x\left(A_y - B_y\right)\right)}}$$

Conclusion,

$$\begin{cases} A(A_x; A_y) \\ B(B_x; B_y) \\ C(C_x; C_y) \\ U(U_x; U_y) - the\ circumcenter\ of\ the\ triangle\ \Delta ABC \end{cases} \rightarrow$$

$$\begin{cases} U_x = \dfrac{(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \\ U_y = \dfrac{(A_x^2 + A_y^2)(C_x - B_x) + (B_x^2 + B_y^2)(A_x - C_x) + (C_x^2 + C_y^2)(B_x - A_x)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \end{cases}$$

(More information: https://en.wikipedia.org/wiki/Circumscribed_circle)

2) Find the coordinates of the orthocenter - $H(H_x; H_y)$

We can use such properties as

$$\overrightarrow{UH} = \overrightarrow{UA} + \overrightarrow{UB} + \overrightarrow{UC}, where\ U - the\ circumcenter\ of\ the\ triangle\ \Delta ABC$$

(More information: https://en.wikipedia.org/wiki/Altitude_(triangle))

In our case,

$$\begin{cases} \overrightarrow{UH} = (H_x - U_x; H_y - U_y) \\ \overrightarrow{UA} = (A_x - U_x; A_y - U_y) \\ \overrightarrow{UB} = (B_x - U_x; B_y - U_y) \\ \overrightarrow{UC} = (C_x - U_x; C_y - U_y) \end{cases} \rightarrow$$

$$H_x - U_x = (A_x - U_x) + (B_x - U_x) + (B_x - U_x) \rightarrow$$

$$\boxed{H_x = (A_x + B_x + C_x) - 2U_x}$$

$$H_y - U_y = (A_y - U_y) + (B_y - U_y) + (B_y - U_y) \rightarrow$$

$$\boxed{H_y = (A_y + B_y + C_y) - 2U_y}$$

As we know

$$\begin{cases} U_x = \dfrac{(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \\ U_y = \dfrac{(A_x^2 + A_y^2)(C_x - B_x) + (B_x^2 + B_y^2)(A_x - C_x) + (C_x^2 + C_y^2)(B_x - A_x)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \end{cases}$$

Then,

$$H_x = (A_x + B_x + C_x) - 2 \cdot \frac{(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)}$$

$$H_y = (A_y + B_y + C_y) - 2 \cdot \frac{(A_x^2 + A_y^2)(C_x - B_x) + (B_x^2 + B_y^2)(A_x - C_x) + (C_x^2 + C_y^2)(B_x - A_x)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)}$$

Or

$$\boxed{H_x = (A_x + B_x + C_x) - \frac{(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)}{\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)}}$$

$$\boxed{H_y = (A_y + B_y + C_y) - \frac{(A_x^2 + A_y^2)(C_x - B_x) + (B_x^2 + B_y^2)(A_x - C_x) + (C_x^2 + C_y^2)(B_x - A_x)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)}}$$

Conclusion,

$$\begin{cases} A(A_x; A_y) \\ B(B_x; B_y) \\ C(C_x; C_y) \\ H(H_x; H_y) - the\ orthocenter\ of\ the\ triangle\ \Delta ABC \end{cases} \to$$

$$\begin{cases} H_x = (A_x + B_x + C_x) - \dfrac{(A_x^2 + A_y^2)(B_y - C_y) + (B_x^2 + B_y^2)(C_y - A_y) + (C_x^2 + C_y^2)(A_y - B_y)}{\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \\ H_y = (A_y + B_y + C_y) - \dfrac{(A_x^2 + A_y^2)(C_x - B_x) + (B_x^2 + B_y^2)(A_x - C_x) + (C_x^2 + C_y^2)(B_x - A_x)}{2\left(A_x(B_y - C_y) + B_x(C_y - A_y) + C_x(A_y - B_y)\right)} \end{cases}$$