Answer on Question #71065 – Math – Calculus

Question

How to find the double point of the curve $x^3 + y^3 - 3axy = 0$?

Solution

A singular point of the curve defined by the equation $F(x, y) = 0$ is a point $M_0(x_0, y_0)$ at which both partial derivatives of the function $F(x, y)$ vanish:

$$\left(\frac{\partial F}{\partial x}\right)_0 = \left(\frac{\partial F}{\partial y}\right)_0 = 0$$

If the second partial derivatives of $F(x, y)$ are not all equal to zero at M_0, then M_0 is called a double point. If all the first and second derivatives vanish at M_0, but the third derivatives are not all equal to zero, then M_0 is called a triple point, and so on. When the nature of a curve near a double point is investigated, an important role is played by the sign of the expression (Hessian matrix)

$$Hessian(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial F}{\partial x \partial y} \\ \frac{\partial F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}$$

$$\Delta = \left(\frac{\partial^2 F}{\partial x^2}\right)_0 \left(\frac{\partial^2 F}{\partial y^2}\right)_0 - \left(\left(\frac{\partial F}{\partial x \partial y}\right)_0\right)^2$$

If $\Delta > 0$, the singular point is isolated (or an acnode).

If $\Delta < 0$, the singular point is a node, or a point of self-intersection (or a crunode, a loop).

If $\Delta = 0$, the singular point is a cusp (or a spinode).

$x^3 + y^3 - 3axy = 0$

$F(x, y) = x^3 + y^3 - 3axy$

$F(x, y) = 0$

$$\frac{\partial F}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial F}{\partial y} = 3y^2 - 3ax$$

$$\begin{cases} F(x, y) = 0 \\ \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} x^3 + y^3 - 3axy = 0 \\ 3x^2 - 3ay = 0 \\ 3y^2 - 3ax = 0 \end{cases}$$

$$\frac{\partial^2 F}{\partial x^2} = 6x$$

$$\frac{\partial^2 F}{\partial y^2} = 6y$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = -3a$$

If $a = 0$

$x^3 + y^3 = 0$

$$\begin{cases} x^3 + y^3 = 0 \\ 3x^2 = 0 \\ 3y^2 = 0 \end{cases} \Rightarrow Point(0,0)$$

$$\frac{\partial^2 F}{\partial x^2} \Big|_{(0,0)} = 0$$

$$\frac{\partial^2 F}{\partial y^2} \Big|_{(0,0)} = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} \Big|_{(0,0)} = 0$$

$$\frac{\partial^3 F}{\partial x^3} = 6$$

$$\frac{\partial^3 F}{\partial y^3} = 6$$

We have that all the first and second derivatives vanish at $(0,0)$, but the third derivatives are not all equal to zero, and then $(0,0)$ is the triple point.

If $a \neq 0$

$$\begin{cases} x^3 + y^3 - 3axy = 0 \\ 3x^2 - 3ay = 0 \\ 3y^2 - 3ax = 0 \end{cases} \Rightarrow \begin{cases} x^3 + y^3 - 3axy = 0 \\ x^2 = ay \\ y^2 = ax \end{cases} \Rightarrow \begin{cases} x^3 + \frac{x^6}{a^3} - 3ax\frac{x^2}{a} = 0 \\ y = \frac{x^2}{a} \\ y^2 = ax \end{cases}$$

$$\begin{cases} \frac{x^6}{a^3} - 2x^3 = 0 \\ y = \frac{x^2}{a} \\ \frac{x^4}{a^2} = ax \end{cases} \Rightarrow \begin{cases} x^3(x^3 - 2a^3) = 0 \\ y = \frac{x^2}{a} \\ x(x^3 - a^3) = 0 \end{cases}$$

We have that

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow Point(0,0)$$

Or

$$\begin{cases} x^3 - 2a^3 = 0 \\ y = \frac{x^2}{a} \\ x^3 - a^3 = 0 \end{cases} \Rightarrow \begin{cases} x^3 - a^3 - a^3 = 0 \\ y = \frac{x^2}{a} \\ x^3 - a^3 = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ y = \frac{x^2}{a} \\ x^3 - a^3 = 0 \end{cases} \Rightarrow No\ solutions$$

$$\frac{\partial^2 F}{\partial x^2}\Big|_{(0,0)} = 0$$
$$\frac{\partial^2 F}{\partial y^2}\Big|_{(0,0)} = 0$$
$$\frac{\partial^2 F}{\partial x \partial y}\Big|_{(0,0)} = -3a \neq 0$$
$$Hessian(F) = \begin{pmatrix} 0 & -3a \\ -3a & 0 \end{pmatrix}$$
$$\Delta = \begin{vmatrix} 0 & -3a \\ -3a & 0 \end{vmatrix} = -9a^2 < 0$$

$\Delta < 0$, the singular point $(0, 0)$ is a node, or a point of self-intersection (or a crunode, a loop).

Answer:
If $a = 0$, there are no double points.
If $a \neq 0$, the singular point $(0, 0)$ is a node or a point of self-intersection (or a crunode, a loop).

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