Answer on Question #70335 – Math – Differential Equations

Question

Solve

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x .$$

Solution

Consider the equation

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = 0 .$$

The fundamental system of solutions of this equation is $y_1 = e^{-i\sqrt{6}x}\sec x$ and $y_2 = -\frac{1}{2\sqrt{6}}e^{i\sqrt{6}x}\sec x$ because

$$\frac{y_1}{y_2} = \frac{e^{-i\sqrt{6}x}\sec x}{-\frac{1}{2\sqrt{6}}e^{i\sqrt{6}x}\sec x} = -2\sqrt{6}e^{-2i\sqrt{6}x} \neq 0$$

and so functions y_1 and y_2 are linearly independent.

Therefore the general solution of the homogeneous equation is

$$y = C_1y_1 + C_2y_2 = C_1e^{-i\sqrt{6}x}\sec x - C_2\frac{1}{2\sqrt{6}}e^{i\sqrt{6}x}\sec x .$$

We show that $y = \frac{1}{7}e^x\sec x$ is a particular solution of the inhomogeneous equation.

$$y' = \frac{1}{7}e^x\sec x(1+\tan x),$$

$$y'' = \frac{e^x}{7}\left(\sec x + 2\sec x\tan x + \sec^3 x + \sec x\tan^2 x\right).$$

So,

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = y'' - 2\tan xy' + 5y =$$

$$= \frac{e^x}{7}\left(\sec x + 2\sec x\tan x + \sec^3 x + \sec x\tan^2 x\right) - \frac{2}{7}e^x\tan x\sec x(1+\tan x) + \frac{5}{7}e^x\sec x =$$

$$= \frac{e^x}{7}\left(\sec x + 2\sec x \tan x + \sec^3 x + \sec x \tan^2 x - 2\tan x \sec x\left(1+\tan x\right) + 5\sec x\right) =$$

$$= \frac{e^x}{7}\left(\sec x + \sec^3 x + \sec x \tan^2 x - 2\tan^2 x \sec x + 5\sec x\right) =$$

$$= \frac{e^x}{7}\left(\sec x + \sec^3 x - \tan^2 x \sec x + 5\sec x\right) = \frac{e^x}{7}\left(\sec x + \sec x(\sec^2 x - \tan^2 x) + 5\sec x\right) =$$

$$= \frac{e^x}{7}\left(\sec x + \sec x + 5\sec x\right) = e^x \sec x .$$

Hence,

$$y = C_1 e^{-i\sqrt{6}x}\sec x - C_2\frac{1}{2\sqrt{6}}e^{i\sqrt{6}x}\sec x + \frac{1}{7}e^x \sec x$$

is the general solution of the inhomogeneous equation.

Answer:

$$y = C_1 e^{-i\sqrt{6}x}\sec x - C_2\frac{1}{2\sqrt{6}}e^{i\sqrt{6}x}\sec x + \frac{1}{7}e^x \sec x$$