Answer on Question #70334 – Math – Differential Equations

Question

The differential equation of a damped vibrating system under the action of an external periodic force is

$$\frac{d^2x}{dt^2} + 2m_0\frac{dx}{dt} + n^2x = a\cos pt$$

Show that, if $n > m_0 > 0$ the complementary function of the differential equation represents vibrations which are soon damped out. Find the particular integral in terms of periodic functions.

Solution

The characteristic equation is

$$\lambda^2 + 2m_0\lambda + n^2 = 0,$$

Its roots are

$$\lambda = \frac{-2m_0 \pm \sqrt{4m_0^2 - 4n^2}}{2} = -m_0 \pm \sqrt{m_0^2 - n^2}$$

If $n > m_0 > 0$, then the characteristic equation has the complex roots, and the complementary function of the differential equation is

$$x = e^{-m_0t}\left(c_1\cos\left(t\sqrt{n^2 - m_0^2}\right) + c_2\sin\left(t\sqrt{n^2 - m_0^2}\right)\right)$$

The multiplier e^{-m_0t} means that vibrations are soon damped out since $m_0 > 0$.

The particular integral:

$$\tilde{x} = A\cos pt + B\sin pt$$

Then

$$\tilde{x}' = -Ap\sin pt + Bp\cos pt$$

$$\tilde{x}'' = -Ap^2\cos pt - Bp^2\sin pt$$

$$\tilde{x}'' + 2m_0\tilde{x}' + n^2\tilde{x} = -Ap^2\cos pt - Bp^2\sin pt + 2m_0(-Ap\sin pt + Bp\cos pt) + n^2(A\cos pt + B\sin pt) =$$

$$= a\cos pt$$

$$\begin{cases} An^2 - Ap^2 - 2m_0Bp = a \\ Bn^2 - Bp^2 - 2m_0Ap = 0 \end{cases}$$

$$A = \frac{B}{2m_0p}(n^2 - p^2)$$

$$\frac{B}{2m_0p}(n^2 - p^2)^2 - 2m_0Bp = a$$

$$B = \frac{2am_0p}{(n^2 - p^2)^2 - 4m_0^2p^2}$$

$$A = \frac{a}{(n^2 - p^2)^2 - 4m_0^2p^2}(n^2 - p^2)$$

Answer:

$$\tilde{x} = \frac{a}{(n^2 - p^2)^2 - 4m_0^2p^2}(n^2 - p^2)\cos pt + \frac{2am_0p}{(n^2 - p^2)^2 - 4m_0^2p^2}\sin pt$$