ANSWER on Question #70313 Math. Calculus

If

$$y(x) = \operatorname{atan} x$$

Obtain an equation showing the relationship between $y^{(n+2)}, y^{(n+1)}$ and $y^{(n)}$

SOLUTION

As we know

$$y(x) = \operatorname{atan} x \leftrightarrow y^{(1)}(x) \equiv y'(x) = \frac{d(\operatorname{atan} x)}{dx} = \frac{1}{1+x^2}$$

$$y^{(2)}(x) \equiv y''(x) = \frac{d}{dx}\left(y'(x)\right) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2} = -\frac{2x}{1+x^2} \cdot \frac{1}{1+x^2} \leftrightarrow$$

$$y^{(2)}(x) = -\frac{2x}{1+x^2} \cdot y^{(1)}(x)\Big| \times (1+x^2) \leftrightarrow (1+x^2) \cdot y^{(2)}(x) = -2x \cdot y^{(1)}(x)$$

$$\boxed{(1+x^2) \cdot y^{(2)}(x) + 2x \cdot y^{(1)}(x) = 0}$$

From the resulting equation we derive the necessary recurrence relation.

We recall that the General Leibniz rule has the form

(More information: https://en.wikipedia.org/wiki/General_Leibniz_rule)

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

What do we do? From the resulting equation we take the nth derivative

$$(1+x^2) \cdot y^{(2)}(x) + 2x \cdot y^{(1)}(x) = 0 \Big| \times \frac{d^n}{dx^n}$$

$$\left((1+x^2) \cdot y^{(2)}(x) + 2x \cdot y^{(1)}(x)\right)^{(n)} = (0)^{(n)}$$

$$\left((1+x^2) \cdot y^{(2)}(x)\right)^{(n)} + \left(2x \cdot y^{(1)}(x)\right)^{(n)} = 0$$

We analyze each derivative separately

$$\left((1+x^2)\cdot y^{(2)}(x)\right)^{(n)} = \left[\begin{array}{c} u = y^{(2)}(x) \\ v = 1+x^2 \\ (u\cdot v)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)} \end{array}\right] =$$

$$= \binom{n}{0}\left(y^{(2)}(x)\right)^{(n)}\cdot(1+x^2)^{(0)} + \binom{n}{1}\left(y^{(2)}(x)\right)^{(n-1)}\cdot(1+x^2)^{(1)} +$$

$$+ \binom{n}{2}\left(y^{(2)}(x)\right)^{(n-2)}\cdot(1+x^2)^{(2)} + \binom{n}{3}\left(y^{(2)}(x)\right)^{(n-3)}\cdot(1+x^2)^{(3)} + \cdots$$

$$\begin{cases} (1+x^2)^{(0)} = 1+x^2 \\ (1+x^2)^{(1)} = 2x \\ (1+x^2)^{(2)} = 2 \\ (1+x^2)^{(3)} = 0 \\ \forall k \geq 3,\ \ (1+x^2)^{(k)} = 0 \end{cases}$$

$$\begin{cases} \binom{n}{0} = \dfrac{n!}{(n-0)!\cdot(0)!} = 1 \\ \binom{n}{1} = \dfrac{n!}{(n-1)!\cdot(1)!} = \dfrac{(n-1)!\cdot n}{(n-1)!} = n \\ \binom{n}{2} = \dfrac{n!}{(n-2)!\cdot(2)!} = \dfrac{(n-2)!\cdot(n-1)\cdot n}{(n-2)!\cdot 2} = \dfrac{n(n-1)}{2} \end{cases}$$

Then,

$$\left((1+x^2)\cdot y^{(2)}(x)\right)^{(n)} = y^{(n+2)}(x)\cdot(1+x^2) + n\cdot y^{(n+1)}(x)\cdot 2x + \frac{n(n-1)}{2}\cdot y^{(n)}(x)\cdot 2$$

$$\boxed{\left((1+x^2)\cdot y^{(2)}(x)\right)^{(n)} = (1+x^2)y^{(n+2)}(x) + 2nx\cdot y^{(n+1)}(x) + n(n-1)\cdot y^{(n)}(x)}$$

$$\left(2x\cdot y^{(1)}(x)\right)^{(n)}=\left[\begin{array}{c} u=y^{(1)}(x)\\ v=2x\\ (u\cdot v)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)}\end{array}\right]=$$

$$=\binom{n}{0}\left(y^{(1)}(x)\right)^{(n)}\cdot(2x)^{(0)}+\binom{n}{1}\left(y^{(1)}(x)\right)^{(n-1)}\cdot(2x)^{(1)}+$$

$$+\binom{n}{2}\left(y^{(1)}(x)\right)^{(n-2)}\cdot(2x)^{(2)}+\binom{n}{3}\left(y^{(1)}(x)\right)^{(n-3)}\cdot(2x)^{(3)}+\cdots$$

$$\begin{cases}(2x)^{(0)}=2x\\ (2x)^{(1)}=2\\ (2x)^{(2)}=0\\ \forall k\geq 3,\ \ (1+x^2)^{(k)}=0\end{cases}$$

$$\begin{cases}\binom{n}{0}=\dfrac{n!}{(n-0)!\cdot(0)!}=1\\ \binom{n}{1}=\dfrac{n!}{(n-1)!\cdot(1)!}=\dfrac{(n-1)!\cdot n}{(n-1)!}=n\\ \binom{n}{2}=\dfrac{n!}{(n-2)!\cdot(2)!}=\dfrac{(n-2)!\cdot(n-1)\cdot n}{(n-2)!\cdot 2}=\dfrac{n(n-1)}{2}\end{cases}$$

Then,

$$\left((2x)\cdot y^{(1)}(x)\right)^{(n)}=y^{(n+1)}(x)\cdot(2x)+n\cdot y^{(n)}(x)\cdot 2+\frac{n(n-1)}{2}\cdot y^{(n)}(x)\cdot 0$$

$$\boxed{\left((2x)\cdot y^{(1)}(x)\right)^{(n)}=2x\,y^{(n+1)}(x)+2n\,y^{(n)}(x)}$$

Conclusion,

$$\left((1+x^2)\cdot y^{(2)}(x)\right)^{(n)}+\left(2x\cdot y^{(1)}(x)\right)^{(n)}=0\leftrightarrow$$

$$(1+x^2)y^{(n+2)}(x)+2nx\cdot y^{(n+1)}(x)+n(n-1)\cdot y^{(n)}(x)+2x\,y^{(n+1)}(x)+2n\,y^{(n)}(x)=0$$

$$(1+x^2)y^{(n+2)}(x)+2nx\cdot y^{(n+1)}(x)+n(n-1)\cdot y^{(n)}(x)+2x\,y^{(n+1)}(x)+2n\,y^{(n)}(x)=0$$

$$(1+x^2)y^{(n+2)}(x)+(2nx+2x)\cdot y^{(n+1)}(x)+[n(n-1)+2n]\cdot y^{(n)}(x)=0$$

$$(1+x^2)y^{(n+2)}(x)+2x(n+1)\cdot y^{(n+1)}(x)+[n^2-n+2n]\cdot y^{(n)}(x)=0$$

$$(1+x^2)y^{(n+2)}(x)+2x(n+1)\cdot y^{(n+1)}(x)+[n^2+n]\cdot y^{(n)}(x)=0$$

$$\boxed{(1+x^2)y^{(n+2)}(x) + 2x(n+1)\cdot y^{(n+1)}(x) + n(n+1)\cdot y^{(n)}(x) = 0}$$

ANSWER

$$y(x) = \operatorname{atan} x \leftrightarrow (1+x^2)y^{(n+2)}(x) + 2x(n+1)\cdot y^{(n+1)}(x) + n(n+1)\cdot y^{(n)}(x) = 0$$