Answer on Question #70256 – Math – Differential Equations

Question

Solve the given initial value problem. Give the largest interval I over which the solution is defined.

$$(1+t^2)\frac{dx}{dt}+x=\tan^{-1}t\,, x(0)=4$$

[Hint: In your solution let $u=\tan^{-1}t$]

Solution

$$(1+t^2)\frac{dx}{dt}+x=\tan^{-1}t$$

This linear first order ODE can be re-arranged to give the following standard form

$$\frac{dx}{dt}+\frac{1}{1+t^2}x=\frac{\tan^{-1}t}{1+t^2}$$

where $P(t)=\dfrac{1}{1+t^2}$ and $Q(t)=\dfrac{\tan^{-1}t}{1+t^2}$

The ODE can be solved using the integrating factor method.

$$IF=e^{\int P(t)dt}=e^{\int\frac{1}{1+t^2}dt}=e^{\tan^{-1}t}$$

This factor is defined so that the equation becomes equivalent to:

$$\frac{d}{dt}(IF\cdot x)=IF\cdot Q(x)$$

Integrating both sides with respect to t, gives:

$$x=\frac{1}{IF}\int IF\cdot Q(x)\,dt$$

Then

$$x=\frac{1}{e^{\tan^{-1}t}}\int e^{\tan^{-1}t}\cdot\frac{\tan^{-1}t}{1+t^2}dt$$

Let $u=\tan^{-1}t$. Then

$$du=\frac{1}{1+t^2}dt$$

$$\int e^{\tan^{-1}t}\cdot\frac{\tan^{-1}t}{1+t^2}dt=\int e^u u\,du=ue^u-\int e^u\,du=ue^u-e^u+C=$$

$$=e^{\tan^{-1}t}\tan^{-1}t-e^{\tan^{-1}t}+C$$

$$\int w\,dv=wv-\int v\,dw$$

$$w=u, dw=du$$

$$dv=e^u du, v=e^u$$

$$x=\frac{1}{e^{\tan^{-1}t}}(e^{\tan^{-1}t}\tan^{-1}t-e^{\tan^{-1}t}+C)$$

$$x = \tan^{-1} t - 1 + \frac{C}{e^{\tan^{-1} t}}$$

$$x(0) = 4 => \tan^{-1}(0) - 1 + \frac{C}{e^{\tan^{-1}(0)}} = 4$$

$$0 - 1 + \frac{C}{e^{0}} = 4$$

$$C = 5$$

Then

$$x = \tan^{-1} t - 1 + \frac{5}{e^{\tan^{-1} t}}$$

The solution is defined on $(-\infty, \infty)$.

Answer: $x = \tan^{-1} t - 1 + \frac{5}{e^{\tan^{-1} t}}, t \in (-\infty, \infty)$.