Answer on Question # 70221 – Math – Calculus

Question

Evaluate integral x⁴/(1+x²)⁴ dx from 0 to ∞ using a reduction formula for the product of trigonometric functions.

Solution

We have

$$\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\,dx$$

To evaluate this integral we substitute $x = \tan\theta$. Then we have

$$(1+x^2)^4 = (1+\tan^2\theta)^4 = (\sec^2\theta)^4 = \sec^8\theta$$

$$dx = d(\tan\theta) = \sec^2\theta d\theta$$

To change the limits of integration we note that when $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = \infty$, $\tan\theta = \infty$, so $\theta = \frac{\pi}{2}$. Thus we have

$$\int_0^{\infty} \frac{x^4}{(1+x^2)^4}\,dx = \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta}\sec^2\theta d\theta = \int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta(1-\sin^2\theta)d\theta = \int_0^{\pi/2} \sin^4\theta d\theta - \int_0^{\pi/2} \sin^6\theta d\theta$$

Use the reduction formula

$$\int \sin^n\theta d\theta = -\frac{1}{n}\cos\theta\sin^{n-1}\theta + \frac{n-1}{n}\int \sin^{n-2}\theta d\theta$$

For the first term $(n = 4)$ we have

$$\int_0^{\pi/2} \sin^4\theta d\theta = -\frac{1}{4}\cos\theta\sin^3\theta\Big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \sin^2\theta d\theta$$

$$= -\frac{1}{4}\left(\cos\frac{\pi}{2}\sin^3\frac{\pi}{2} - \cos0\sin^3 0\right) + \frac{3}{4}\left(-\frac{1}{2}\cos\theta\sin\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} d\theta\right)$$

$$= 0 + \frac{3}{4}\left(-\frac{1}{2}\left(\cos\frac{\pi}{2}\sin\frac{\pi}{2} - \cos0\sin0\right) + \frac{1}{2}\theta\Big|_0^{\pi/2}\right) =$$

$$= \frac{3}{4}\left(0 + \frac{1}{2}\left(\frac{\pi}{2} - 0\right)\right) = \frac{3\pi}{16}$$

For the second term $(n = 6)$ we have

$$\int_0^{\pi/2} \sin^6\theta d\theta = -\frac{1}{6}\cos\theta \sin^5\theta\Big|_0^{\pi/2} + \frac{5}{6}\int_0^{\pi/2} \sin^4\theta d\theta$$

However

$$\int_0^{\pi/2} \sin^4\theta d\theta = \frac{3\pi}{16}$$

so we get

$$= -\frac{1}{6}\left(\cos\frac{\pi}{2}\sin^5\frac{\pi}{2} - \cos 0 \sin^5 0\right) + \frac{5}{6}\cdot\frac{3\pi}{16} = 0 + \frac{15\pi}{96} = \frac{15\pi}{96}$$

Finally we get

$$\int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta = \int_0^{\pi/2} \sin^4\theta d\theta - \int_0^{\pi/2} \sin^6\theta d\theta = \frac{3\pi}{16} - \frac{15\pi}{96} = \frac{3\pi}{96} = \frac{\pi}{32}$$

Answer:

$$\int_0^{\infty} \frac{x^4}{(1+x^2)^4} dx = \frac{\pi}{32}$$