Answer on Question #70102 - Math – Calculus

Question

Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$

Solution

If $f(x, y)$ is continuous on a region D such that

$$D = \{(x, y) | a \le x \le b, \quad g_1(x) \le y \le g_2(x)\}$$

then [1]

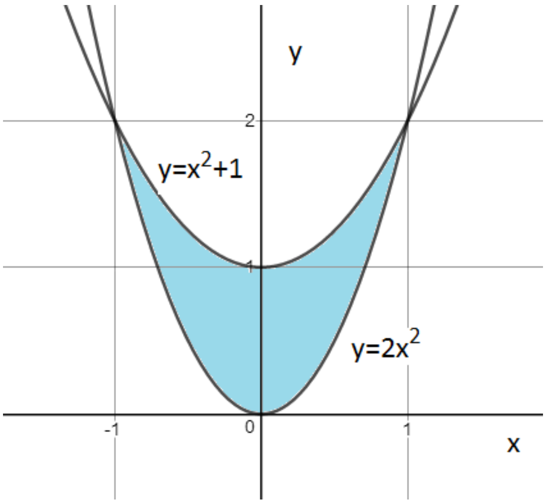
$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

We have $g_1(x) = 2x^2$ is the lower boundary and $g_2(x) = 1 + x^2$ is the upper boundary.

Find a and b. Given parabolas intersects when $2x^2 = 1 + x^2$ or $x^2 = 1$, so $x = \pm 1$ and $y = 2 \cdot (\pm 1)^2 = 2$

The region D is $D = \{(x, y) | -1 \le x \le 1, \ 2x^2 \le y \le 1 + x^2\}$

Sketch the region D



Then we have

$$\iint_D f(x, y)dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)dydx$$

Find this integral

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)dydx = \int_{-1}^{1} (xy + y^2)|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} (x(1+x^2) + (1+x^2)^2) - (x(2x^2) + (2x^2)^2)dx$$

$$= \int_{-1}^{1} \left((x + x^3 + 1 + 2x^2 + x^4) - (2x^3 + 4x^4)\right)dx$$

$$= \int_{-1}^{1} (1 + x + 2x^2 - x^3 - 3x^4)dx = \left(x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5}\right)\Bigg|_{-1}^{1}$$

$$= \left(1 + \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{3}{5}\right) - \left(-1 + \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{3}{5}\right) = 2 + \frac{4}{3} - \frac{6}{5} = \frac{32}{15}$$

Answer: $\iint_D (x + 2y)dA = \frac{32}{15}$

References:

1. James Stewart, Calculus, Seventh Edition [2012], p.1014.