ANSWER on Question #70041 Math. Calculus

Determine the square of the arc element for the curvilinear coordinate system (u, v, w) whose coordinates are related to the Cartesian coordinates as follows:

$$\begin{cases} x = 3u + v - w \\ y = u + 2v + 2w \\ z = 2u - v - w \end{cases}$$

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As we know, in the Cartesian coordinates

$$\vec{r} = x \cdot \overrightarrow{e_x} + y \cdot \overrightarrow{e_y} + z \cdot \overrightarrow{e_z}$$

$$d\vec{r} = dx \cdot \overrightarrow{e_x} + dy \cdot \overrightarrow{e_y} + dz \cdot \overrightarrow{e_z}$$

$$\boxed{(dr)^2 = d\vec{r} \cdot d\vec{r} \equiv (dx)^2 + (dy)^2 + (dz)^2}$$

In the curvilinear coordinate system (u, v, w)

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} \cdot du + \frac{\partial \vec{r}}{\partial v} \cdot dv + \frac{\partial \vec{r}}{\partial w} \cdot dw \equiv \overrightarrow{h_u} \cdot du + \overrightarrow{h_v} \cdot dv + \overrightarrow{h_w} \cdot dw$$

$$\boxed{(dr)^2 = d\vec{r} \cdot d\vec{r} \equiv \left(\overrightarrow{h_u}\right)^2 (du)^2 + \left(\overrightarrow{h_v}\right)^2 (dv)^2 + \left(\overrightarrow{h_w}\right)^2 (dw)^2}$$

Where

$$\overrightarrow{h_u} = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial u} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial u} \cdot \overrightarrow{e_z}$$

$$\overrightarrow{h_v} = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial v} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial v} \cdot \overrightarrow{e_z}$$

$$\overrightarrow{h_w} = \frac{\partial \vec{r}}{\partial w} = \frac{\partial x}{\partial w} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial w} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial w} \cdot \overrightarrow{e_z}$$

In our case,

$$\begin{cases} x = 3u + v - w \\ y = u + 2v + 2w \\ z = 2u - v - w \end{cases}$$

Then,

$$\overrightarrow{h_u} = \frac{\partial x}{\partial u} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial u} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial u} \cdot \overrightarrow{e_z} =$$

$$= \frac{\partial(3u + v - w)}{\partial u} \cdot \overrightarrow{e_x} + \frac{\partial(u + 2v + 2w)}{\partial u} \cdot \overrightarrow{e_y} + \frac{\partial(2u - v - w)}{\partial u} \cdot \overrightarrow{e_z} =$$

$$= 3 \cdot \overrightarrow{e_x} + 1 \cdot \overrightarrow{e_y} + 2 \cdot \overrightarrow{e_z}$$

$$\overrightarrow{h_u} = 3 \cdot \overrightarrow{e_x} + 1 \cdot \overrightarrow{e_y} + 2 \cdot \overrightarrow{e_z} \leftrightarrow \left(\overrightarrow{h_u}\right)^2 = 3^2 + 1^2 + 2^2 = 9 + 1 + 4 = 14$$

$$\boxed{\left(\overrightarrow{h_u}\right)^2 = 14}$$

$$\overrightarrow{h_v} = \frac{\partial x}{\partial v} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial v} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial w} \cdot \overrightarrow{e_z} =$$

$$= \frac{\partial(3u + v - w)}{\partial v} \cdot \overrightarrow{e_x} + \frac{\partial(u + 2v + 2w)}{\partial v} \cdot \overrightarrow{e_y} + \frac{\partial(2u - v - w)}{\partial w} \cdot \overrightarrow{e_z} =$$

$$= 1 \cdot \overrightarrow{e_x} + 2 \cdot \overrightarrow{e_y} - 1 \cdot \overrightarrow{e_z}$$

$$\overrightarrow{h_v} = 1 \cdot \overrightarrow{e_x} + 2 \cdot \overrightarrow{e_y} - 1 \cdot \overrightarrow{e_z} \leftrightarrow \left(\overrightarrow{h_v}\right)^2 = 1^2 + 2^2 + (-1)^2 = 1 + 4 + 1 = 6$$

$$\boxed{\left(\overrightarrow{h_v}\right)^2 = 6}$$

$$\overrightarrow{h_w} = \frac{\partial x}{\partial w} \cdot \overrightarrow{e_x} + \frac{\partial y}{\partial w} \cdot \overrightarrow{e_y} + \frac{\partial z}{\partial w} \cdot \overrightarrow{e_z} =$$

$$= \frac{\partial(3u + v - w)}{\partial w} \cdot \overrightarrow{e_x} + \frac{\partial(u + 2v + 2w)}{\partial w} \cdot \overrightarrow{e_y} + \frac{\partial(2u - v - w)}{\partial w} \cdot \overrightarrow{e_z} =$$

$$= (-1) \cdot \overrightarrow{e_x} + 2 \cdot \overrightarrow{e_y} - 1 \cdot \overrightarrow{e_z}$$

$$\overrightarrow{h_u} = (-1) \cdot \overrightarrow{e_x} + 2 \cdot \overrightarrow{e_y} - 1 \cdot \overrightarrow{e_z} \leftrightarrow \left(\overrightarrow{h_u}\right)^2 = (-1)^2 + 2^2 + (-1)^2 = 1 + 4 + 1 = 6$$

$$\boxed{\left(\overrightarrow{h_u}\right)^2 = 6}$$

Conclusion,

$$(dr)^2 \equiv \left(\overrightarrow{h_u}\right)^2 (du)^2 + \left(\overrightarrow{h_v}\right)^2 (dv)^2 + \left(\overrightarrow{h_w}\right)^2 (dw)^2 = 14(du)^2 + 6(dv)^2 + 6(dw)^2$$

$$\boxed{(dr)^2 \equiv 14(du)^2 + 6(dv)^2 + 6(dw)^2}$$

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$$(dr)^2 \equiv 14(du)^2 + 6(dv)^2 + 6(dw)^2$$