ANSWER on Question #70040 – Math – Calculus

QUESTION

Obtain the curl of the following vector field:

$$\vec{A}(\rho;\phi;z) = \rho^3 \cdot \overrightarrow{e_\rho} + \rho z \cdot \overrightarrow{e_\phi} + \rho z \sin\phi \cdot \overrightarrow{e_z}$$

SOLUTION

As we can see, the vector field $\vec{A}(\rho;\phi;z)$ is given in cylindrical coordinates.

By the definition, the curl in cylindrical coordinates has the form

$$\vec{\nabla} \times \vec{A}(\rho;\phi;z) =$$

$$= \overrightarrow{e_\rho} \cdot \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \overrightarrow{e_\phi} \cdot \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \overrightarrow{e_z} \cdot \left(\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$

(More information: https://en.wikipedia.org/wiki/Cylindrical_coordinate_system)

In this case,

$$\vec{A}(\rho;\phi;z) = \rho^3 \cdot \overrightarrow{e_\rho} + \rho z \cdot \overrightarrow{e_\phi} + \rho z \sin\phi \cdot \overrightarrow{e_z} \leftrightarrow \begin{cases} A_\rho = \rho^3 \\ A_\phi = \rho z \\ A_z = \rho z \sin\phi \end{cases}$$

We calculate the bracket for each vector separately.

1) $\overrightarrow{e_\rho}$:

$$\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} = \frac{1}{\rho}\frac{\partial(\rho z \sin\phi)}{\partial \phi} - \frac{\partial(\rho z)}{\partial z} = \frac{1}{\rho} \cdot \rho z \cos\phi - \rho = z\cos\phi - \rho$$

$$\boxed{\overrightarrow{e_\rho} \cdot \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) = \overrightarrow{e_\rho} \cdot (z\cos\phi - \rho)}$$

2) $\overrightarrow{e_\phi}$:

$$\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = \frac{\partial(\rho^3)}{\partial z} - \frac{\partial(\rho z \sin\phi)}{\partial \rho} = 0 - z\sin\phi = -z\sin\phi$$

$$\boxed{\overrightarrow{e_\phi} \cdot \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) = \overrightarrow{e_\phi} \cdot (-z\sin\phi)}$$

3) $\overrightarrow{e_z}$:

$$\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi} = \frac{1}{\rho}\frac{\partial(\rho\cdot\rho z)}{\partial \rho} - \frac{1}{\rho}\frac{\partial(\rho^3)}{\partial \phi} = \frac{1}{\rho}\frac{\partial(\rho^2 z)}{\partial \rho} - \frac{1}{\rho}\cdot 0 = \frac{1}{\rho}\cdot 2\rho z - 0 = 2z$$

$$\boxed{\overrightarrow{e_z} \cdot \left(\frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right) = \overrightarrow{e_z} \cdot (2z)}$$

Conclusion:

$$\boxed{\begin{cases} \vec{A}(\rho;\phi;z) = \rho^3\cdot\overrightarrow{e_\rho} + \rho z\cdot\overrightarrow{e_\phi} + \rho z\sin\phi\cdot\overrightarrow{e_z} \\ \vec{\nabla}\times\vec{A}(\rho;\phi;z) = \overrightarrow{e_\rho}\cdot(z\cos\phi - \rho) + \overrightarrow{e_\phi}\cdot(-z\sin\phi) + \overrightarrow{e_z}\cdot(2z) \end{cases}}$$

Answer: $\vec{\nabla}\times\vec{A}(\rho;\phi;z) = \overrightarrow{e_\rho}\cdot(z\cos\phi - \rho) + \overrightarrow{e_\phi}\cdot(-z\sin\phi) + \overrightarrow{e_z}\cdot(2z)$.