<u>ANSWER on Question #70039 – Math – Calculus</u>

QUESTION

Show that for any scalar field $\phi(r)$:

$$\vec{\nabla} \times \left(\phi(r)\vec{\nabla}\big(\phi(r)\big)\right) = \vec{0}$$

SOLUTION

By the definition, the nabla operator has the form

$$\vec{\nabla} = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Then,

$$\vec{r} = \vec{\imath}\,x + \vec{\jmath}\,y + \vec{k}\,z$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2y}{2\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2z}{2\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}$$

$$\vec{\nabla}\big(\phi(r)\big) = \vec{\imath}\frac{\partial\big(\phi(r)\big)}{\partial x} + \vec{\jmath}\frac{\partial\big(\phi(r)\big)}{\partial y} + \vec{k}\frac{\partial\big(\phi(r)\big)}{\partial z} =$$

$$= \vec{\imath}\frac{\partial\big(\phi(r)\big)}{\partial r}\cdot\frac{\partial r}{\partial x} + \vec{\jmath}\frac{\partial\big(\phi(r)\big)}{\partial r}\cdot\frac{\partial r}{\partial y} + \vec{k}\frac{\partial\big(\phi(r)\big)}{\partial r}\cdot\frac{\partial r}{\partial z} = \vec{\imath}\phi'(r)\frac{x}{r} + \vec{\jmath}\phi'(r)\frac{y}{r} + \vec{k}\phi'(r)\frac{z}{r}$$

$$= \frac{\phi'(r)}{r}\cdot\left(\vec{\imath}\,x + \vec{\jmath}\,y + \vec{k}\,z\right) = \phi'(r)\cdot\frac{\vec{r}}{r}$$

Conclusion,

$$\boxed{\vec{\nabla}\big(\phi(r)\big) = \phi'(r) \cdot \frac{\vec{r}}{r}}$$

$$\phi(r)\vec{\nabla}\big(\phi(r)\big) = \phi(r) \cdot \left(\phi'(r) \cdot \frac{\vec{r}}{r}\right) = \phi(r) \cdot \phi'(r) \cdot \frac{\vec{r}}{r}$$

$$\boxed{\phi(r)\vec{\nabla}\big(\phi(r)\big) = \phi(r) \cdot \phi'(r) \cdot \frac{\vec{r}}{r}}$$

By the definition, the $\vec{\nabla}\times$ operator has the form

$$\vec{\nabla}\times\ \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In our case,

$$\vec{\nabla}\times\left(\phi(r)\vec{\nabla}\big(\phi(r)\big)\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi(r) \cdot \phi'(r) \cdot \frac{x}{r} & \phi(r) \cdot \phi'(r) \cdot \frac{y}{r} & \phi(r) \cdot \phi'(r) \cdot \frac{z}{r} \end{vmatrix} =$$

$$= \vec{i} \cdot \left(\frac{\partial}{\partial y}\left[\phi(r) \cdot \phi'(r) \cdot \frac{z}{r}\right] - \frac{\partial}{\partial z}\left[\phi(r) \cdot \phi'(r) \cdot \frac{y}{r}\right]\right) -$$

$$-\vec{j} \cdot \left(\frac{\partial}{\partial x}\left[\phi(r) \cdot \phi'(r) \cdot \frac{z}{r}\right] - \frac{\partial}{\partial z}\left[\phi(r) \cdot \phi'(r) \cdot \frac{x}{r}\right]\right) +$$

$$+\vec{k} \cdot \left(\frac{\partial}{\partial x}\left[\phi(r) \cdot \phi'(r) \cdot \frac{y}{r}\right] - \frac{\partial}{\partial y}\left[\phi(r) \cdot \phi'(r) \cdot \frac{x}{r}\right]\right)$$

Notice, that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=-\frac{1}{2}\cdot\frac{2x}{\sqrt{(x^2+y^2+z^2)^3}}=-\frac{x}{r^3}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right)=\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=-\frac{1}{2}\cdot\frac{2y}{\sqrt{(x^2+y^2+z^2)^3}}=-\frac{y}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right)=\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=-\frac{1}{2}\cdot\frac{2z}{\sqrt{(x^2+y^2+z^2)^3}}=-\frac{z}{r^3}$$

We consider each bracket separately

1)

$$\frac{\partial}{\partial y}\left[\phi(r)\cdot\phi'(r)\cdot\frac{z}{r}\right]-\frac{\partial}{\partial z}\left[\phi(r)\cdot\phi'(r)\cdot\frac{y}{r}\right]=$$

$$=\phi'(r)\cdot\frac{y}{r}\cdot\phi'(r)\cdot\frac{z}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{y}{r}\cdot\frac{z}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{y}{r^3}\right)\cdot\frac{z}{r}-$$

$$-\left[\phi'(r)\cdot\frac{z}{r}\cdot\phi'(r)\cdot\frac{y}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{z}{r}\cdot\frac{y}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{z}{r^3}\right)\cdot\frac{y}{r}\right]=$$

$$=\phi'(r)\cdot\phi'(r)\cdot\frac{zy}{r^2}+\phi(r)\cdot\phi''(r)\cdot\frac{zy}{r^2}-\phi(r)\cdot\phi'(r)\cdot\frac{zy}{r^4}-$$

$$-\phi'(r)\cdot\phi'(r)\cdot\frac{zy}{r^2}-\phi(r)\cdot\phi''(r)\cdot\frac{zy}{r^2}+\phi(r)\cdot\phi'(r)\cdot\frac{zy}{r^4}=0$$

2)

$$\frac{\partial}{\partial x}\left[\phi(r)\cdot\phi'(r)\cdot\frac{z}{r}\right]-\frac{\partial}{\partial z}\left[\phi(r)\cdot\phi'(r)\cdot\frac{x}{r}\right]=$$

$$=\phi'(r)\cdot\frac{x}{r}\cdot\phi'(r)\cdot\frac{z}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{x}{r}\cdot\frac{z}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{x}{r^3}\right)\cdot\frac{z}{r}-$$

$$-\left[\phi'(r)\cdot\frac{z}{r}\cdot\phi'(r)\cdot\frac{x}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{z}{r}\cdot\frac{x}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{z}{r^3}\right)\cdot\frac{x}{r}\right]=$$

$$=\phi'(r)\cdot\phi'(r)\cdot\frac{xz}{r^2}+\phi(r)\cdot\phi''(r)\cdot\frac{xz}{r^2}-\phi(r)\cdot\phi'(r)\cdot\frac{xz}{r^4}-$$

$$-\phi'(r)\cdot\phi'(r)\cdot\frac{xz}{r^2}-\phi(r)\cdot\phi''(r)\cdot\frac{xz}{r^2}+\phi(r)\cdot\phi'(r)\cdot\frac{xz}{r^4}=0$$

3)

$$\frac{\partial}{\partial x}\left[\phi(r)\cdot\phi'(r)\cdot\frac{y}{r}\right]-\frac{\partial}{\partial y}\left[\phi(r)\cdot\phi'(r)\cdot\frac{x}{r}\right]=$$

$$=\phi'(r)\cdot\frac{x}{r}\cdot\phi'(r)\cdot\frac{y}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{x}{r}\cdot\frac{y}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{x}{r^3}\right)\cdot\frac{y}{r}-$$

$$-\left[\phi'(r)\cdot\frac{y}{r}\cdot\phi'(r)\cdot\frac{x}{r}+\phi(r)\cdot\phi''(r)\cdot\frac{y}{r}\cdot\frac{x}{r}+\phi(r)\cdot\phi'(r)\cdot\left(-\frac{y}{r^3}\right)\cdot\frac{x}{r}\right]=$$

$$=\phi'(r)\cdot\phi'(r)\cdot\frac{xy}{r^2}+\phi(r)\cdot\phi''(r)\cdot\frac{xy}{r^2}-\phi(r)\cdot\phi'(r)\cdot\frac{xy}{r^4}-$$

$$-\phi'(r)\cdot\phi'(r)\cdot\frac{xy}{r^2}-\phi(r)\cdot\phi''(r)\cdot\frac{xy}{r^2}+\phi(r)\cdot\phi'(r)\cdot\frac{xy}{r^4}=0$$

Conclusion,

$$\vec{\nabla}\times\left(\phi(r)\vec{\nabla}\big(\phi(r)\big)\right)=\vec{\imath}\cdot\left(\frac{\partial}{\partial y}\left[\phi(r)\cdot\phi'(r)\cdot\frac{z}{r}\right]-\frac{\partial}{\partial z}\left[\phi(r)\cdot\phi'(r)\cdot\frac{y}{r}\right]\right)-$$

$$-\vec{\jmath}\cdot\left(\frac{\partial}{\partial x}\left[\phi(r)\cdot\phi'(r)\cdot\frac{z}{r}\right]-\frac{\partial}{\partial z}\left[\phi(r)\cdot\phi'(r)\cdot\frac{x}{r}\right]\right)+$$

$$+\vec{k}\cdot\left(\frac{\partial}{\partial x}\left[\phi(r)\cdot\phi'(r)\cdot\frac{y}{r}\right]-\frac{\partial}{\partial y}\left[\phi(r)\cdot\phi'(r)\cdot\frac{x}{r}\right]\right)=$$

$$=\vec{\imath}\cdot 0-\vec{\jmath}\cdot 0+\vec{k}\cdot 0=\vec{0}$$

$$\boxed{\vec{\nabla}\times\left(\phi(r)\vec{\nabla}\big(\phi(r)\big)\right)=\vec{0}}$$

Q.E.D

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