Answer on Question #70017 – Math – Differential Equations

Question

Use the power series method to obtain one solution of the following ODE:

$$x^2y'' + 3y' - xy = 0$$

Solution

Assume that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution. Then we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}, x^2y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}$$

Substituting for x^2y'', y' and xy in the given differential equation, we obtain the following series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 3\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 3\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 3(1)a_1x^0 + 3(2)a_2x^1 + \sum_{n=2}^{\infty} 3(n+1)a_{n+1}x^n - a_0x^1 -$$

$$-\sum_{n=2}^{\infty} a_{n-1}x^n = 0$$

$$3a_1 = 0 => a_1 = 0$$

$$6a_2 - a_0 = 0 => a_2 = \frac{a_0}{6}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=2}^{\infty} (-1)a_{n-1}x^n = 0$$

$$\sum_{n=2}^{\infty} x^n[n(n-1)a_n + 3(n+1)a_{n+1} - a_{n-1}] = 0$$

$n(n-1)a_n + 3(n+1)a_{n+1} - a_{n-1} = 0, n = 2, 3, 4, \ldots$

$n = 2$

$2a_2 + 9a_3 - a_1 = 0 \Rightarrow 9a_3 = -2a_2 \Rightarrow a_3 = -\frac{2}{9}a_2 = -\frac{2}{9}\left(\frac{a_0}{6}\right) = -\frac{a_0}{27}$

$n = 3$

$6a_3 + 12a_4 - a_2 = 0 \Rightarrow 12a_4 = -6a_3 + a_2 \Rightarrow a_4 = \frac{-6\left(-\frac{a_0}{27}\right) + \frac{a_0}{6}}{12} = \frac{7a_0}{216}$

$n = 4$

$12a_4 + 15a_5 - a_3 = 0 \Rightarrow 15a_5 = -12a_4 + a_3 \Rightarrow a_5 = \frac{-12\left(\frac{7a_0}{216}\right) - \frac{a_0}{27}}{15} =$

$= -\frac{23a_0}{810}$

$n = 5$

$20a_5 + 18a_6 - a_4 = 0 \Rightarrow 18a_6 = -20a_5 + a_4 \Rightarrow$

$\Rightarrow a_6 = \frac{-20\left(-\frac{23a_0}{810}\right) + \frac{7a_0}{216}}{18} = \frac{389a_0}{11664}$

$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$

$y(x) = a_0 + 0 + \frac{a_0}{6}x^2 - \frac{a_0}{27}x^3 + \frac{7a_0}{216}x^4 - \frac{23a_0}{810}x^5 + \frac{389a_0}{11664}x^6 + \cdots =$

$= a_0\left(1 + \frac{x^2}{6} - \frac{x^3}{27} + \frac{7x^4}{216} - \frac{23x^5}{810} + \frac{389x^6}{11664} + \cdots\right)$

$y_1(x) = 1 + \frac{x^2}{6} - \frac{x^3}{27} + \frac{7x^4}{216} - \frac{23x^5}{810} + \frac{389x^6}{11664} + \cdots$

$y_2(x) = 0$

Answer: $y_1(x) = 1 + \frac{x^2}{6} - \frac{x^3}{27} + \frac{7x^4}{216} - \frac{23x^5}{810} + \frac{389x^6}{11664} + \cdots$