Answer on Question #70006 – Math – Calculus

QUESTION

Obtain a unit tangent vector to any point on the curve defined by the parametric equations:

$$x = sin\ 3t, \quad y = 2cos\ 3t, \quad z = 4t.$$

SOLUTION

By the definition, for vector $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$ a unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}, \qquad where\ |\vec{r'}(t)| = \sqrt{(x')^2 + (y')^2 + (z')^2}$$

In our case,

$$\begin{cases} x(t) = \sin 3t \\ y(t) = 2\cos 3t \\ z(t) = 4t \end{cases} \rightarrow \boxed{\begin{cases} x'(t) = 3\cos 3t \\ y'(t) = -2\cdot 3\sin 3t = -6\sin 3t \\ z'(t) = 4 \end{cases}}$$

$$|\vec{r'}(t)| = \sqrt{(3\cos 3t)^2 + (-6\sin 3t)^2 + (4)^2} = \sqrt{9\cos^2 3t + 36\sin^2 3t + 16} =$$

$$= \left| \begin{matrix} \sin^2 t + \cos^2 t = 1 \\ A\sin^2 t + A\cos^2 t = A \end{matrix} \right| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 27\sin^2 3t + 16} =$$

$$= \sqrt{9 + 27\sin^2 3t + 16} = \sqrt{25 + 27\sin^2 3t} = \sqrt{25}\sqrt{1 + \frac{27}{25}\sin^2 3t} = 5\sqrt{1 + \frac{27}{25}\sin^2 3t}$$

$$\boxed{|\vec{r'}(t)| = 5\sqrt{1 + \frac{27}{25}\sin^2 3t}}$$

Then,

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{x'(t)\vec{\imath} + y'(t)\vec{\jmath} + z'(t)\vec{k}}{|\vec{r'}(t)|} =$$

$$= \frac{3\cos 3t\,\vec{\imath} - 6\sin 3t\,\vec{\jmath} + 4\vec{k}}{5\sqrt{1 + \frac{27}{25}\sin^2 3t}} = \frac{1}{\sqrt{1 + \frac{27}{25}\sin^2 3t}} \cdot \left(\frac{3}{5}\cos 3t\,\vec{\imath} - \frac{6}{5}\sin 3t\,\vec{\jmath} + \frac{4}{5}\vec{k}\right)$$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{1 + \frac{27}{25}\sin^2 3t}} \cdot \left(\frac{3}{5}\cos 3t\,\vec{\imath} - \frac{6}{5}\sin 3t\,\vec{\jmath} + \frac{4}{5}\vec{k}\right)}$$

ANSWER:

$$\vec{T}(t) = \frac{1}{\sqrt{1 + \frac{27}{25}\sin^2 3t}} \cdot \left(\frac{3}{5}\cos 3t\,\vec{i} - \frac{6}{5}\sin 3t\,\vec{j} + \frac{4}{5}\vec{k}\right), \forall t \in \mathbb{R}$$