Answer on Question #69886 – Math – Differential Equations

Question

i) The initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \ \ y(0) = 0$$

has a unique solution in some interval of the form $-h < x < h$.

Solution

Theorem (Existence and uniqueness theorem for First-Order ODE's):

> Let $f\,(x, y)$ be a real valued function which is continuous on the rectangle
> $$R = \{(x, y); |x - x_0| \le a, |y - y_0| \le b\}, (a, b > 0).$$
> Assume f has a partial derivative with respect to y and that $\partial f / \partial y$ is also continuous on the rectangle R.
> Then there exists an interval $I = [x_0 - h, x_0 + h]$ with $h \le a$ such that the initial value problem
> $$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$
> has a unique solution $y(x)$ defined on the interval I.

We have that

$$f\,(x, y) = x^2 + y^2, x_0 = 0, y_0 = 0$$

The function f is continuously differentiable in $\mathbb{R}^2$ because its partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = 2x, \ \ \frac{\partial f}{\partial y}(x, y) = 2y$$

are continuous functions in $\mathbb{R}^2$. Therefore, the initial problem

$$\frac{dy}{dx} = x^2 + y^2, \ \ y(0) = 0$$

has a unique solution in an interval

$$-h < x < h, \qquad h > 0.$$

Consider the rectangle

$$R = \{(x, y); |x| \le a, |y| \le b\}, (a, b > 0).$$

Since f is continuous in a closed and bounded domain, it is necessarily bounded in R, i.e., there exists $K > 0$ such that

$$|f(x, y)| \le K, \qquad \forall (x, y) \in R.$$

Set

$$K = \max_{(x,y) \in R} |f(x, y)|.$$

Then the initial value problem has at most one solution $y = y(x)$ in the interval

$$-h < x < h, \ \ h > 0,$$

where

$$h = \min\left\{a, \frac{b}{K}\right\}.$$

We have

$$K = \max_{(x,y)\in R} |f(x,y)| = \max_{(x,y)\in R} (x^2 + y^2) = a^2 + b^2 => h = \min\left\{a, \frac{b}{a^2 + b^2}\right\}$$

Set

$$a = 1$$

and

$$b = 1.$$

Then

$$h = \min\left\{1, \frac{1}{(1)^2 + (1)^2}\right\} = \frac{1}{2}.$$

Hence, we can state that the solution $y = y(x)$ of the initial value problem exists in the interval $-0.5 < x < 0.5$.

Answer: The statement is true. There exists a unique solution $y = y(x)$ of the initial value problem in the interval $-0.5 < x < 0.5$.