Answer on Question #69869 – Math – Calculus

Which of the following statements are true or false? Give reasons for your answers.

Question

i) The function $f: R \to R$, given by $f(x) = \ln\left|x + \sqrt{1 + x^2}\right|$ is neither even nor odd.

Solution

Find the domain if f

$$\begin{cases} 1 + x^2 \geq 0 \\ \left|x + \sqrt{1 + x^2}\right| > 0 \end{cases} => x \in R$$

Domain: $(-\infty, \infty)$

Find

$$f(-x) = \ln\left|(-x) + \sqrt{1 + (-x)^2}\right| = \ln\left|-x + \sqrt{1 + x^2}\right|$$

In general

$\ln\left|-x + \sqrt{1 + x^2}\right| \neq \ln\left|x + \sqrt{1 + x^2}\right|$

$\ln\left|-x + \sqrt{1 + x^2}\right| \neq -\ln\left|x + \sqrt{1 + x^2}\right|$

Example

$x = 1;\ \ln\left|-1 + \sqrt{1 + (1)^2}\right| = \ln\left|-1 + \sqrt{2}\right| = \ln(\sqrt{2} - 1)$

$\ln\left|1 + \sqrt{1 + (1)^2}\right| = \ln\left|1 + \sqrt{2}\right| = \ln(\sqrt{2} + 1)$

Since $\sqrt{2} - 1 \neq \sqrt{2} + 1$, then $\ln(\sqrt{2} - 1) \neq \ln(\sqrt{2} + 1)$

$\exists x \in R, f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Therefore, the function $f: R \to R$, given by $f(x) = \ln\left|x + \sqrt{1 + x^2}\right|$ is neither even nor odd.

Answer: the statement is true.

Question

ii) $\frac{d}{dx}\int_x^0 \sin(t^2)\,dt = -\sin(x^2)$

Solution

According to Properties of the definite integral

$$\int_x^0 \sin(t^2)\,dt = -\int_0^x \sin(t^2)\,dt$$

Then

$$\frac{d}{dx}\int_x^0 \sin(t^2)\,dt = -\frac{d}{dx}\int_0^x \sin(t^2)\,dt$$

The Fundamental Theorem, Part II.

If f is continuous on an open interval I containing the point a, then the function

$$\int_a^x f(t)\,dt$$

is differentiable on I and for all x in I,

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

Therefore

$$\frac{d}{dx}\int_x^0 \sin(t^2)\,dt = -\frac{d}{dx}\int_0^x \sin(t^2)\,dt = -\sin(x^2)$$

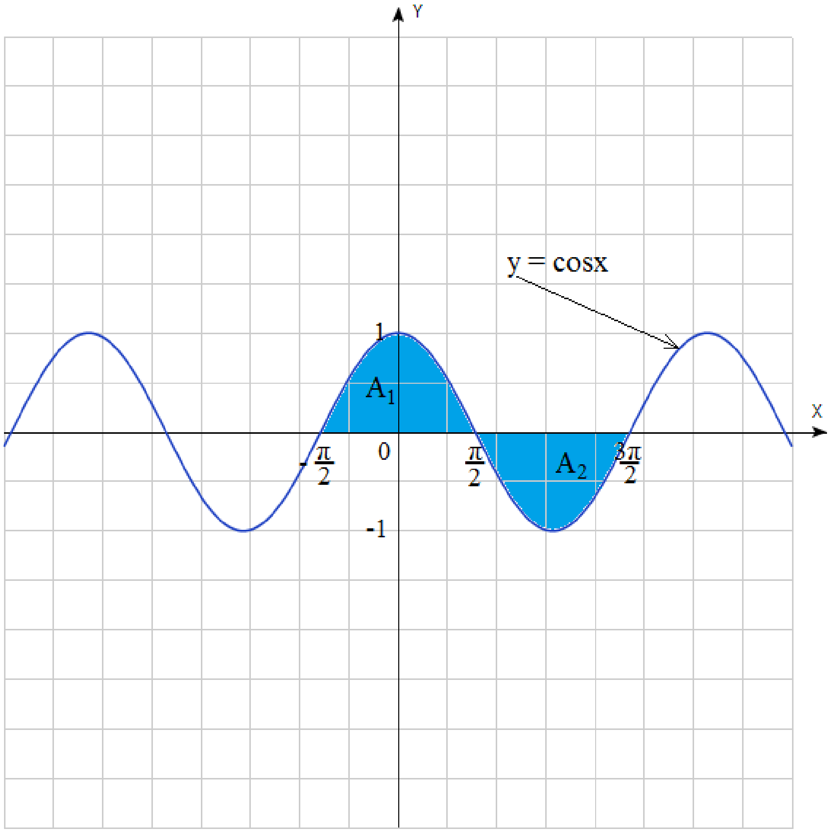
for all x in open interval I containing the point 0.

Answer: the statement is true.

Question

iii) The area enclosed by the x −axis and the curve $y = \cos x$ over the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is 0.

Solution



$$Area = A_1 + A_2 = \int_{-\pi/2}^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\frac{3\pi}{2}} \cos x\, dx =$$

$$= \sin x \left|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \sin x \right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) - \left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) =$$

$$= 1 - (-1) - (-1 - 1) = 4\ (sq.units) \neq 0$$

Answer: the statement is false.

Question

iv) If f and g are functions over R such that $f + g$ is continuous, then f must be continuous.

Solution

Consider the counterexample.
The one-dimensional Heaviside step function centered at 0 is defined in the following way

$$H(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases}$$

The discontinuity is at $x = 0$.
Let $f(x) = H(x)$ and $g(x) = -H(x)$. Then

$$f(x) + g(x) = H(x) + \left(-H(x)\right) = 0, x \in R$$

We have that $f + g$ is continuous on R but f has discontinuity at $x = 0$.
Answer: the statement is false.

Question

v) $x - y + 2 = 0$ is a tangent to the curve $(x + y)^3 = (x - y + 2)^2$ at $(-1, 1)$.

Solution

Let us consider the function

$$F(x, y) = (x + y)^3 - (x - y + 2)^2 = 0$$

We search for the derivative of the implicit function $y = y(x)$ at the point $(-1, 1)$, given by the equation

$$F(x, y) = 0$$

We have

$$\frac{\partial F}{\partial y} = F_y'(x, y) = 3(x + y)^2 - 2(x - y + 2)(-1) = 3(x + y)^2 + 2(x - y + 2)$$

Then

$$F_y'(-1,1) = 3(-1 + 1)^2 + 2(-1 - 1 + 2) = 0$$

Hence $y'(-1)$ is undefined. Therefore, the line $x - y + 2 = 0$ cannot be a tangent to the curve $(x + y)^3 = (x - y + 2)^2$ at $(-1, 1)$.
Answer: the statement is false.