Answer on Question #69748 – Math / Calculus

Find the length of the arc of the hyperbolic spiral $r\theta = a$ lying between $r = a$ and $r = 2a$.

Solution:

The length of the curve is given by formula

$$l = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

In the case a hyperbolic spiral $r(\theta) = \frac{a}{\theta}$ that lying between $r = a$ and $r = 2a$ we have relations

$$\theta_1 = \frac{1}{2}, \qquad \theta_2 = 1$$

$$\frac{dr}{d\theta} = -\frac{a}{\theta_2}, \qquad \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}}.$$

Thus

$$l = \int_{1/2}^{1} \sqrt{\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}}\, d\theta = a \int_{1/2}^{1} \sqrt{1 + \frac{1}{\theta^2}}\, \frac{d\theta}{\theta} = a\left(\sqrt{5} - \sqrt{2} + \operatorname{arcsinh}[1] - \operatorname{arcsinh}\left[\frac{1}{2}\right]\right) = 1.222a$$

Answer: $\left(\sqrt{5} - \sqrt{2} + \operatorname{arcsinh}[1] - \operatorname{arcsinh}\left[\frac{1}{2}\right]\right) a \approx 1.222a.$