Answer on Question #69650 – Math – Differential Equations

Question

Solve the second order differential equation

$$\frac{\partial^2 y}{\partial x^2} - 4y = 12x,\ y(0) = 4,\ y'(0) = 1.$$

Solution

First, let's solve the corresponding homogenous equation $y'' - 4y = 0$. It's linear, so we can solve it using characteristic equation:

$$t^2 - 4 = 0,$$
$$(t+2)(t-2) = 0,$$
$$t_1 = -2, \qquad t_2 = 2$$

There are two distinct eigenvalues, hence the general solution of this homogenous equation is

$$y_0(x) = C_1 e^{t_1 x} + C_2 e^{t_2 x} = C_1 e^{-2x} + C_2 e^{2x}.$$

Second, let's find any particular solution of the original non-homogenous equation. Right-hand side of the initial non-homogeneous differential equation is a polynomial of the first order, $12x$, therefore, we'll look for a particular solution in polynomial form of the same order,

$y_1(x) = ax + b.$

$$y_1 = ax + b, \qquad y_1'' = 0,$$

$$y'' - 4y = 0 - 4(ax + b) = -4ax - b = 12x,$$
$$\begin{cases} -4a = 12 \\ -b = 0 \end{cases} \to \begin{cases} a = -3 \\ b = 0. \end{cases}$$

Thus, $y_1(x) = -3x$ is a particular solution of the non-homogenous equation. Altogether, the general solution of the non-homogenous equation is the sum of any particular solution and the general solution of the corresponding homogenous equation:

$$y(x) = y_0(x) + y_1(x) = C_1 e^{-2x} + C_2 e^{2x} - 3x.$$

Finally, let's apply initial conditions: $y(0) = 4$ and $y'(0) = 1$.

$$y' = (C_1 e^{-2x} + C_2 e^{2x} - 3x)' = -2C_1 e^{-2x} + 2C_2 e^{2x} - 3,$$
$$y'(0) = -2C_1 e^{-2\cdot 0} + 2C_2 e^{2\cdot 0} - 3 = -2C_1 + 2C_2 - 3,$$
$$y(0) = C_1 e^{-2\cdot 0} + C_2 e^{2\cdot 0} - 3\cdot 0 = C_1 + C_2.$$

To do this, we need to solve the system:

$$\begin{cases} C_1 + C_2 = 4 \\ -2C_1 + 2C_2 - 3 = 1 \end{cases} \to \begin{cases} C_2 = 4 - C_1 \\ C_2 = C_1 + 2 \end{cases} \to \begin{cases} C_1 + 2 = 4 - C_1 \\ C_2 = C_1 + 2 \end{cases} \to \begin{cases} C_1 = 1 \\ C_2 = C_1 + 2 \end{cases} \to \begin{cases} C_1 = 1 \\ C_2 = 3. \end{cases}$$

Thus, the solution of the initial value problem is $y(x) = e^{-2x} + 3e^{2x} - 3x$.

<u>Answer:</u> $y(x) = e^{-2x} + 3e^{2x} - 3x$.