Answer on Question #69647 – Math – Differential Equations

Question

Find a particular solution of

$$\frac{dy}{dx} + 2y = 2x^2 + 3\,.$$

Solution

Method 1

It’s a first order linear differential equation, hence it has the following form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

with $P(x) = 2$ and $Q(x) = \ 2x^2 + 3$.

The associated homogeneous differential equation is

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} + 2y = 0$$

The characteristic equation is

$$\lambda + 2 = 0.$$

Its root is

$$\lambda = -2.$$

The solution of the homogeneous equation is

$$y_h = c_1 e^{-2x} \quad (1)$$

The right-hand side of the differential equation

$$\frac{dy}{dx} + 2y = 2x^2 + 3 \quad (2)$$

is a second degree polynomial that does not coincide with (1), therefore, a particular solution of (2) can be found in the following way:

$$y_p = ax^2 + bx + c, \ (3)$$

$$y_p' = 2ax + b. \qquad (4)$$

Substituting formulae (3), (4) into the differential equation (2) one gets

$$\frac{dy_p}{dx} + 2y_p = 2x^2 + 3$$

$$2ax + b + 2(ax^2 + bx + c) = 2x^2 + 3$$

$$2ax^2 + (2b + 2a)x + (b + 2c) = 2x^2 + 3,$$

Equating coefficients before the like terms one gets the system

$$\begin{cases} 2a = 2 \\ 2b + 2a = 0 \\ b + 2c = 3 \end{cases}$$

$$\begin{cases} a = 1 \\ b = -a \\ c = \dfrac{3-b}{2} \end{cases}$$

$$\begin{cases} a = 1 \\ b = -1 \\ c = 2 \end{cases}$$

Thus, a particular solution of (2) is

$$y_p = ax^2 + bx + c = x^2 - x + 2$$

Method 2

A particular solution can be found by means of the general solution of the differential equation.

The first way

is application of an auxiliary function.

Let

$$y = uv,$$

where $u = u(x)$ and $v = v(x)$ are the auxiliary functions.

Then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

So we have

$$\frac{du}{dx}v + u\frac{dv}{dx} + 2uv = 2x^2 + 3$$

or

$$\frac{du}{dx}v + u\left(\frac{dv}{dx} + 2v\right) = 2x^2 + 3. \qquad (1)$$

Since one of the auxiliary functions can be chosen arbitrarily, let's choose a function v such that it will satisfy the condition:

$$\frac{dv}{dx} + 2v = 0.$$

So we have

$$\frac{dv}{dx} + 2v = 0$$

or

$$\frac{dv}{dx} = -2v.$$

It’s a separable equation which can be easily integrated:

$$\int \frac{dv}{v} = -2\int dx$$

$$ln|v| = -2x$$

$$v = e^{-2x}\,.$$

Substitute the expression for the function v into equation (1):

$$\frac{du}{dx}e^{-2x} = 2x^2 + 3$$

or

$$\frac{du}{dx} = (2x^2 + 3)e^{2x}.$$

It’s a separable equation so we can integrate it:

$$\int du = \int (2x^2 + 3)e^{2x}dx.$$

To calculate the second integral we shall use the formula for integration by parts:

$$\int udv = uv - \int vdu.$$

Let

$$u = 2x^2 + 3 \quad and \quad dv = e^{2x}dx,$$

then

$$du = 4xdx; \quad v = \frac{1}{2}e^{2x}\,.$$

Using the formula for integration by parts, we have

$$\int (2x^2 + 3)e^{2x}dx = \frac{(2x^2 + 3)}{2}e^{2x} - 2\int xe^{2x}dx.$$

This integral still contains a product of functions. So we shall use the formula for integration by parts again. This time we choose

$$u = x \ \text{ and } \ dv = e^{2x}dx$$

then

$$du = dx; \quad v = \frac{1}{2}e^{2x}\,.$$

So

$$\int (2x^2 + 3)e^{2x}dx = \frac{(2x^2 + 3)}{2}e^{2x} - 2\int xe^{2x}dx =$$

$$= \frac{(2x^2 + 3)}{2}e^{2x} - 2\left(\frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx\right) = \frac{(2x^2 + 3)}{2}e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + C =$$

$$= (x^2 - x + 2)e^{2x} + C,$$

where C is an integration constant.

So we get the following auxiliary function:

$$u = (x^2 - x + 2)e^{2x} + C.$$

After the back substitution we have the general solution of the given equation:

$$y = x^2 - x + 2 + Ce^{-2x}, \ (2)$$

where C is arbitrary real constant.

A particular solution can be obtained if in (2) we set C to be equal to a certain number. For example, if $C = 0$, then $y_p = x^2 - x + 2$ will be a particular solution.

The second way

is using an integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

The solution is then commonly written as

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx.$$

Determine the integrating factor:

$$\mu(x) = e^{2\int dx} = e^{2x}.$$

Then the solution is

$$y = e^{-2x} \int (2x^2 + 3)e^{2x}\,dx = e^{-2x}\left((x^2 - x + 2)e^{2x} + C\right) = x^2 - x + 2 + Ce^{-2x},$$

where C is an integration constant.

So the general solution of the given equation is

$$y = x^2 - x + 2 + Ce^{-2x}. \ (3)$$

A particular solution can be obtained if in (3) we set C to be equal to a certain number. For example, if $C = 0$, then $y_p = x^2 - x + 2$ will be a particular solution.

Answer: $y_p = x^2 - x + 2.$