Answer on Question #69646, Math / Differential Equations

Solve the differential equation $y' = -y + x^2y^2$

Solution

It's not hard to see that this is a Bernoulli differential equation

$$y' + p(x)y = q(x)y^n$$

In this case, we have that $n = 2$

$$y' + y = x^2y^2.$$

Then to find the solution, change the dependent variable from y to z, where $z = y^{-1}$. This gives a differential equation in x and z that is linear, and can be solved using the integrating factor method.

$$y = \frac{1}{z}, y' = \left(\frac{1}{z}\right)' = -\frac{1}{z^2}z'$$

$$-\frac{1}{z^2}z' + \frac{1}{z} = x^2\frac{1}{z^2}$$

$$-z' + z = x^2$$

$$z' - z = -x^2$$

Integrating Factor $= v(x) = e^{\int P(x)dx} = e^{\int(-1)dx} = e^{-x}$

Constant of integration is 0, so v is simple as possible.

Multiple both sides by a positive function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot z$.

$$e^{-x}\frac{dz}{dx} - e^{-x}z = -x^2e^{-x}$$

$$\frac{d}{dx}(e^{-x}z) = e^{-x}\frac{dz}{dx} - e^{-x}z$$

$$\frac{d}{dx}(e^{-x}z) = -x^2e^{-x}$$

$$\int d(e^{-x}z) = \int(-x^2e^{-x})\,dx$$

$$\int(-x^2e^{-x})\,dx = x^2e^{-x} - 2\int xe^{-x}\,dx = x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C$$

$$\int u\,dv = uv - \int v\,du$$

$u = x^2, du = 2xdx$

$dv = -e^{-x}dx, v = e^{-x}$

$$-2\int xe^{-x}\,dx = 2xe^{-x} - 2\int e^{-x}\,dx = 2xe^{-x} + 2e^{-x} + C$$

$$\int u\,dv = uv - \int v\,du$$

$u = 2x, du = 2dx$

$dv = -e^{-x}dx, v = e^{-x}$

$e^{-x}z = x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C$

$z = x^2 + 2x + 2 + Ce^x$

Put back $z = y^{-1}$

$$y = \frac{1}{x^2 + 2x + 2 + Ce^x}$$

Answer: $y = \dfrac{1}{x^2 + 2x + 2 + Ce^x}$.