Answer on Question #69645 – Math – Differential Equations

Question

Solve the Riccati's equation

$$\frac{dy}{dx} = -1 - x^2 + y^2.$$

Solution

One searches for the general solution in the following form

$$y(x) = z(x) + y_p(x),$$

where $y_p(x)$ is a particular solution of the given equation.

This substitution transforms the equation into Bernoulli's one of the unknown function $z(x)$.

Let the particular solution be

$$y_p(x) = ax.$$

Substituting it into the differential equation we get

$$\frac{d}{dx}(ax) = -1 - x^2 + (ax)^2$$

or

$$a = -1 + (a^2 - 1)x^2.$$

Value

$$a = -1$$

satisfies the previous equation.

The particular solution is

$$y_p(x) = -x,$$

and the general solution may be found as

$$y(x) = z(x) - x.$$

Substituting it into the Riccati's equation we get

$$\frac{d}{dx}(z(x) - x) = -1 - x^2 + (z(x) - x)^2,$$

$$\frac{dz}{dx} - 1 = -1 - x^2 + z^2 - 2xz + x^2,$$

$$\frac{dz}{dx} = z^2 - 2xz.$$

This is Bernoulli's equation. It can be solved using the substitution $z(x) = u(x)v(x)$:

$$\frac{d(uv)}{dx} = (uv)^2 - 2xuv,$$

$$v\frac{du}{dx} + u\frac{dv}{dx} + 2xuv - u^2v^2 = 0,$$

$$\left(\frac{du}{dx} + 2xu\right)v + \left(u\frac{dv}{dx} - u^2v^2\right) = 0,$$

We get the system

$$\begin{cases} \dfrac{du}{dx} + 2xu = 0 \\ u\dfrac{dv}{dx} - u^2v^2 = 0 \end{cases}.$$

We need a particular solution of the first equation of the system. Check that it is $u(x) = be^{-x^2}$:

$$\frac{d\left(be^{-x^2}\right)}{dx} + 2bxe^{-x^2} = 0,$$

$$-2xbe^{-x^2} + 2bxe^{-x^2} = 0,$$

$$0 = 0.$$

Let $b = 1$, so a particular solution is $u(x) = e^{-x^2}$.

As $u(x) = e^{-x^2} \neq 0$, the second equation may be reduced to

$$\frac{dv}{dx} - uv^2 = 0$$

or

$$\frac{dv}{dx} - e^{-x^2}v^2 = 0.$$

This is the separable DE.

$$\int \frac{dv}{v^2} = \int e^{-x^2}dx,$$

$$-\frac{1}{v(x)} = \frac{\sqrt{\pi}}{2}\operatorname{erf} x - \frac{C}{2},$$

where C is an integration constant.

Then

$$v(x) = -\frac{1}{\frac{\sqrt{\pi}}{2}\operatorname{erf} x - \frac{C}{2}} = \frac{2}{C - \sqrt{\pi}\operatorname{erf} x}$$

and

$$z(x) = u(x)v(x) = \frac{2e^{-x^2}}{C - \sqrt{\pi}\operatorname{erf} x}.$$

Finally, the general solution of the Riccati's equation is

$$y(x) = z(x) - x = \frac{2e^{-x^2}}{C - \sqrt{\pi}\operatorname{erf} x} - x.$$

Answer: $y(x) = \frac{2e^{-x^2}}{C - \sqrt{\pi}\operatorname{erf} x} - x.$