Answer on Question #69636 – Math – Differential Equations

Question

Find the general solution of the differential equation

$$y' = \frac{y}{x} + \frac{y^3}{x^3}.$$

Solution

This is a Bernoulli equation for $n = 3$. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{1}{xy^2} + \frac{1}{x^3}.$$

Substitute the function $v = 1/y^2$ and its derivative $v' = -2(y'/y^3)$, into the differential equation above:

$$-\frac{v'}{2} = \frac{v}{x} + \frac{1}{x^3} \;\Rightarrow\; v' = -\frac{2}{x}v - \frac{2}{x^3} \;\Rightarrow\; v' + \frac{2}{x}v = -\frac{2}{x^3}.$$

The last equation is a linear differential equation for the function v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$, then

$$(x^2v)' = -\frac{2}{x} \;\Rightarrow\; x^2v = -2\ln x + C.$$

We obtain that

$$v = \frac{-2\ln x + C}{x^2}.$$

Since $v = 1/y^2$,

$$\frac{1}{y^2} = \frac{-2\ln x + C}{x^2} \;\Rightarrow y = \pm\frac{x}{\sqrt{-2\ln x + C}}.$$

Answer: $y = \pm\frac{x}{\sqrt{-2\ln x + C}}$.