Answer on Question #69635 – Math – Differential Equations

Question

Solve the differential equation

$$(x^2 + y^2)dx - 2xydy = 0 .$$

Solution

Rewrite the differential equation as

$$y' = \frac{x^2 + y^2}{2xy} .$$

This is Euler homogeneous equation, since

$$f(cx, cy) = \frac{c^2x^2 + c^2y^2}{2(cx)(cy)} = \frac{c^2(x^2 + y^2)}{c^2(2xy)} = \frac{x^2 + y^2}{2xy} = f(x, y) .$$

Since the numerator and denominator are homogeneous "degree 2" we multiply the right-hand side of the equation by "1" in the form $(1/x^2)/(1/x^2)$:

$$y' = \frac{x^2 + y^2}{2xy} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} .$$

Now we introduce the change of functions $v = y/x$,

$$y' = \frac{1 + v^2}{2v}.$$

Since $y = xv$, then $y' = v + xv'$, which implies

$$v + xv' = \frac{1 + v^2}{2v} \quad \Rightarrow \quad xv' = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v} = \frac{1 - v^2}{2v}.$$

We obtained the separable equation

$$v' = \frac{1}{x}\left(\frac{1 - v^2}{2v}\right).$$

We rewrite and integrate it,

$$-\frac{2v}{v^2 - 1}v' = \frac{1}{x} \quad \Rightarrow -\int \frac{2v}{v^2 - 1}dv = \int \frac{1}{x}dx + C_0.$$

The substitution $u = v^2(x) - 1$ implies $du = 2v(x)dv = 2v(x)v'(x)dx$, so

$$-\int \frac{du}{u} = \int \frac{dx}{x} + C_0 \ \Rightarrow \ -\ln u = \ln x + C_0 \ \Rightarrow \ u = \frac{1}{e^{\ln x + C_0}} .$$

But $u = 1/e^{\ln x}e^{C_0}$, so denoting $C_1 = e^{C_0}$, then $u = 1/(C_1 x)$. So, we get

$$v^2 - 1 = \frac{1}{C_1 x} \;\Rightarrow\; \left(\frac{y}{x}\right)^2 - 1 = \frac{1}{C_1 x} \;\Rightarrow\; \left(\frac{y}{x}\right)^2 = \frac{1 + C_1 x}{C_1 x} \;\Rightarrow\; y^2 = \frac{1}{C_1} x + x^2$$

or

$$y = \pm\sqrt{C_2 x + x^2}. \text{ (1)}$$

If $v = 1$, then $y = x$, $y' = 1$ and

$y' = \frac{x^2+y^2}{2xy} \Longrightarrow 1 = 1,$

Which is true, hence $y = x$ is a solution that can be obtained from the general solution (1) if $C_2 = 0$.

If $v = -1$, then $y = -x$, $y' = -1$ and

$y' = \frac{x^2+y^2}{2xy} \Longrightarrow -1 = -1,$

which is true, hence $y = -x$ is a solution that can be obtained from the general solution (1) if $C_2 = 0$.

Answer: $y = \pm\sqrt{C_2 x + x^2}$.