Answer on Question #69632 – Math – Differential Equations

Question

Solve the equation

$$x\frac{dy}{dx} - ay = x + 1, (1)$$

where a is a constant.

Solution

A First Order Linear Nonhomogeneous DE can be solved with a help of the Method of Integrating Factor.
Divide both sides of (1) by x

$$\frac{dy}{dx} - \frac{a}{x}y = 1 + \frac{1}{x} \ (2)$$

Integrating factor is

$$v(x) = e^{\int\left(-\frac{a}{x}\right)dx},$$

where

$$\int\left(-\frac{a}{x}\right)dx = -a\ln|x| + C.$$

Set the constant of integration C to be equal to 0 in order to simplify v and $|x| = x$ if $x > 0$.
Then

$$v(x) = e^{\int\left(-\frac{a}{x}\right)dx} = e^{-a\ln x} = \left(e^{\ln x}\right)^{-a} = x^{-a}$$

Multiple both sides of (2) by a positive function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$.

$$x^{-a}\frac{dy}{dx} - x^{-a}\frac{a}{x}y = x^{-a}\left(1 + \frac{1}{x}\right)$$

$$\frac{d}{dx}(x^{-a}y) = x^{-a} + x^{-a-1}$$

<u>*Case*</u> $a = 0$:

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$dy = \left(1 + \frac{1}{x}\right)dx$$

$$\int dy = \int\left(1 + \frac{1}{x}\right)dx$$

$$y = x + \ln|x| + C_1$$

<u>*Case*</u> $a = 1$:

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{1}{x}$$

$$\frac{d}{dx}(x^{-1}y) = x^{-1} + x^{-1-1}$$
$$\int d(x^{-1}y) = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$$
$$\frac{y}{x} = \ln|x| - \frac{1}{x} + C_2$$
$$y = x\ln|x| - 1 + C_2 x$$

<u>Case</u> $a \neq 0, a \neq 1$:

$$\frac{d}{dx}(x^{-a}y) = x^{-a} + x^{-a-1}$$

$$\int (x^{-a}y) = \int (x^{-a} + x^{-a-1})\, dx$$
$$x^{-a}y = \frac{x^{-a+1}}{-a+1} + \frac{x^{-a}}{-a} + C_3$$
$$y = \frac{x}{-a+1} - \frac{1}{a} + C_3 x^a$$

Answer:

$y = \frac{x}{-a+1} - \frac{1}{a} + C_3 x^a$, if $a \neq 0, a \neq 1$;

$y = x + \ln|x| + C_1$, if $a = 0$;

$y = x\ln|x| - 1 + C_2 x$, if $a = 1$.