Answer on Question #69630 – Math – Differential Equations

Question

1. Find the general solution of $\frac{dy}{dx} = (2y^2 + 3xy)/x^2$.

Solution

$$\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2} \Rightarrow y' = \frac{3}{x}y + \frac{2}{x^2}y^2 \Rightarrow$$

$$y' - \frac{3}{x}y = \frac{2}{x^2}y^2. \quad \textbf{(1)}$$

Method 1

Equation (1) is a first-order linear differential equation.

1. Let's make the substitution: $y = UV$
2. Then $y' = U'V + UV'$.
3. Let's put the substitution into the differential equation (1):

$$U'V + UV' - \frac{3}{x}UV = 2x^{-2}U^2V^2.$$

$$U'V + U\left(V' - \frac{3}{x}V\right) = 2x^{-2}U^2V^2.$$

$$\begin{cases} V' - \frac{3}{x}V = 0 \\ U'V = 2x^{-2}U^2V^2 \end{cases} \quad \textbf{(2)}$$

4. Let's solve the first differential equation of the system (2):

$$V' - \frac{3}{x}V = 0.$$

$$\frac{dV}{dx} = \frac{3}{x}V.$$

$$\frac{dV}{V} = \frac{3}{x}dx, V \neq 0$$

$$\int \frac{dV}{V} = \int \frac{3}{x}dx.$$

$$\ln|V| = 3\ln|x| + C.$$

$$V = Cx^3.$$

Put $C = 1$, then

$$V = x^3.$$

5. Let's solve the second differential equation of the system (2):

$$U'V = 2x^{-2}U^2V^2 \Rightarrow U' = 2x^{-2}U^2V \Rightarrow U' = 2x^{-2}U^2x^3 \Rightarrow U' = 2xU^2.$$

Then

$$\frac{dU}{U^2} = 2xdx, U \neq 0 \Rightarrow \int \frac{dU}{U^2} = \int 2xdx \Rightarrow -\frac{1}{U} = x^2 + C \Rightarrow U = -\frac{1}{x^2 + C}$$

6. The general solution of the differential equation (1):

$$y = UV = \left(-\frac{1}{x^2 + C}\right)x^3 = -\frac{x^3}{x^2 + C}.$$

If $U = 0$ or $V = 0$ then $y = 0$ is the special solution of the differential equation (1).

Method 2

The equation (1) is homogeneous first-order differential equation because $\frac{3\lambda y}{\lambda x} = \frac{3y}{x}$ and $2\frac{(\lambda y)^2}{(\lambda x)^2} = 2\frac{y^2}{x^2}$.

1. Let's make the substitution $y = ux$, where u is a new variable,
2. then $y' = (ux)' = u'x + u$.
3. Let's put the substitution into the differential equation (1):

$$u'x + u - 3u = 2u^2,$$

$$x\frac{du}{dx} = 2u^2 + 2u,$$

$$x\frac{du}{dx} = 2u(u+1),$$

$$\frac{du}{u(u+1)} = 2\frac{dx}{x}, u \neq 0, u \neq -1$$

$$\left(\frac{1}{u} - \frac{1}{u+1}\right) du = 2\frac{dx}{x},$$

$$\int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = 2\int \frac{dx}{x},$$

$$\ln|u| - \ln|u+1| = 2\ln x + \ln|C_1|,$$

$$\ln\frac{|u|}{|u+1|} = \ln|C_1|x^2$$

$$\frac{u}{u+1} = C_2x^2,$$

$$\frac{u+1}{u+1} - \frac{1}{u+1} = C_2x^2,$$

$$\frac{1}{u+1} = 1 - C_2x^2,$$

$$u + 1 = \frac{1}{1 - C_2x^2}$$

$$u = \frac{1}{1 - C_2x^2} - 1$$

$$u = \frac{1 - 1 + C_2x^2}{1 - C_2x^2},$$

$$u = -\frac{x^2}{x^2 - \frac{1}{C_2}}.$$

4. The general solution of the differential equation (1):

$$\frac{y}{x} = -\frac{x^2}{x^2+C},$$

$$y = -\frac{x^3}{x^2+C}. \textbf{ (3)}$$

If $u = 0$, then $y = 0$ is the special solution of the differential equation (1).

If $u = -1$, then $y = -x$ is a solution of the differential equation (1) which is a particular case of (3) if we put $C = 0$.

Answer:

$$y = -\frac{x^3}{x^2 + C} \text{ is the general solution;}$$

$$y = 0 \text{ is the special solution.}$$