Answer on Question #69629 – Math – Differential Equations

Question

Solve the initial value problem $(1+y^2)dx+(1+x^2)dy=0, y(0)=-1$.

Solution

$$(1+y^2)dx+(1+x^2)dy=0, \quad (1)$$
$$y(0)=-1.$$

The equation (1) is separable because dividing (1) by $(1+y^2)(1+x^2)$ the variables can be separated:

$$\frac{dx}{1+x^2}+\frac{dy}{1+y^2}=0,$$
$$\frac{dy}{1+y^2}=-\frac{dx}{1+x^2}.$$

Integrating both sides we get

$$\int\frac{dy}{1+y^2}=\int-\frac{dx}{1+x^2},$$
$$\tan^{-1}y=-\tan^{-1}x+C.$$

When $x=0,\ y(0)=-1$:

$$\tan^{-1}(-1)=-\tan^{-1}(0)+C,$$
$$-\frac{\pi}{4}=0+C,$$
$$C=-\frac{\pi}{4},$$
$$\tan^{-1}y=-\tan^{-1}x-\frac{\pi}{4}. \qquad (2)$$

Take the tangent of both sides of equality (2):

$$\tan(\tan^{-1}y)=\tan\left(-\tan^{-1}x-\frac{\pi}{4}\right),$$
$$y=-\tan\left(\tan^{-1}x+\frac{\pi}{4}\right),$$
$$y=-\frac{\tan(\tan^{-1}x)+\tan\left(\frac{\pi}{4}\right)}{1-\tan(\tan^{-1}x)\cdot\tan\left(\frac{\pi}{4}\right)},$$
$$y=-\frac{x+1}{1-x},$$
$$y=\frac{x+1}{x-1}.$$

Answer: $y=\dfrac{x+1}{x-1}$.