Answer on Question #69627 – Math – Differential Equations

Question

Find the general solution of

$$y' + 2y = x^3 e^{-2x}.$$

Solution

It's a first order linear differential equation, that is, it has the following form:

$$y' + P(x)y = Q(x)$$

with $P(x) = 2$ and $Q(x) = x^3 e^{-2x}$.

It can be integrated in different ways.

The first way is using auxiliary functions.

Let

$$y = uv, \quad (1)$$

where $u = u(x)$ and $v = v(x)$ are the auxiliary functions.

Then

$$y' = u'v + uv'.$$

So we have

$$u'v + uv' + 2uv = x^3 e^{-2x}$$

or

$$u'v + u(v' + 2v) = x^3 e^{-2x}. \quad (2)$$

Since one of the auxiliary functions can be chosen arbitrarily, let a function v be such that it will satisfy the following condition:

$$v' + 2v = 0. \quad (3)$$

So we have

$$\frac{dv}{dx} + 2v = 0$$

or

$$\frac{dv}{dx} = -2v.$$

It's a separable equation that can be easily integrated:

$$\int \frac{dv}{v} = -2 \int dx$$

$$ln|v| = -2x$$

$$v = e^{-2x}. (4)$$

Substitute the expression (4) for the function v into equation (2) using (3):

$$u'e^{-2x} = x^3e^{-2x}$$

or

$$u' = x^3$$

$$\frac{du}{dx} = x^3.$$

It's a separable equation so we can integrate it:

$$\int du = \int x^3 dx$$

$$u = \frac{1}{4}x^4 + C, \quad (5)$$

where C is an integration constant.

After substitution (4), (5) back into (1) we have a general solution of the given equation:

$$y = \left(\frac{1}{4}x^4 + C\right)e^{-2x},$$

where C is an integration constant.

The second way is using an integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

The solution is then commonly written as

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx.$$

Determine the integrating factor:

$$\mu(x) = e^{2\int dx} = e^{2x}.$$

Then the solution is

$$y = e^{-2x}\int e^{2x}x^3e^{-2x}dx = e^{-2x}\int x^3dx = e^{-2x}\left(\frac{1}{4}x^4 + C\right).$$

So the general solution of the given equation is

$$y = \left(\frac{1}{4}x^4 + C\right)e^{-2x},$$

where C is an integration constant.

Answer: $y = \left(\frac{1}{4}x^4 + C\right)e^{-2x}$.