Answer on Question #69107 – Math – Real Analysis

Question

Check whether the following sequences{sn} are Cauchy, where
(i) sn = 1+2+3+....+n ;
(ii) sn = [4n^(3)+3n]/[3n^(3)+n^(2)] ;

Solution

Definition. Sequence $\{s_n\}$ is Cauchy if

$$\forall\varepsilon > 0\ \exists N\ \forall n, m > N: |s_n - s_m| < \varepsilon$$

(i) First method.
For the sequence $s_n = 1 + 2 + 3 + \cdots + n$, $n \geq 1$, the smallest difference between its two elements is 1 ($s_{n+1} - s_n = 1$). It cannot be less than any $\varepsilon > 0$ by the definition.
So this sequence is not Cauchy.

Second method.
Every real Cauchy sequence is convergent, but the sequence $s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ does not converge $\left(\lim_{n\to\infty} \frac{n(n+1)}{2} \neq 0\right)$, so this sequence is not Cauchy.

(ii) We have the sequence $s_n = \frac{4n^3+3n}{3n^3+n^2} = \frac{4n^2+3}{3n^2+n}$.
Its limit is

$$\lim_{n\to\infty} \frac{4n^2+3}{3n^2+n} = \lim_{n\to\infty} \frac{4+\frac{3}{n^2}}{3+\frac{1}{n}} = \frac{4}{3}.$$

By the definition of a convergent sequence, let $\varepsilon > 0$. Choose N so that if $n > N$, then

$$\left|s_n - \frac{4}{3}\right| < \varepsilon$$

Now let's check if this sequence is Cauchy:

$$\forall \frac{\varepsilon}{2} > 0\ \exists N\ \forall n, m > N: |s_n - s_m| < \frac{\varepsilon}{2};$$

$$|s_n - s_m| = \left|s_n - \frac{4}{3} - \left(s_m - \frac{4}{3}\right)\right| \leq \left|s_n - \frac{4}{3}\right| + \left|s_m - \frac{4}{3}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence this sequence is Cauchy.