Answer on Question 68975 - Math - Differential Equations

Find the temperature in a bar of length L with both ends insulated and with initial temperature in the rod being $\sin \frac{\pi x}{L}$.

Solution: We consider the initial boundary value problem for the heat equation

$$u_t = a^2 u_{xx}, \qquad x \in (0, L), \quad t > 0, \tag{1}$$

$$u(0, x) = \sin \frac{\pi x}{L}, \tag{2}$$

$$u_x(t, 0) = u_x(t, L) = 0. \tag{3}$$

Let us find a *special type* of non-trivial solutions

$$v(t, x) = T(t)X(x) \tag{4}$$

of (1) that satisfies conditions (3). We will show that there exists a countable set of such solutions

$$v_n(t, x) = T_n(t)X_n(x), \quad n = 0, 1, 2, \ldots$$

Then we will look for a solution u of (1)–(3) in the form

$$u(t, x) = \sum_{n=0}^{\infty} v_n(t, x).$$

We substitute (4) into the differential equation (1), and divide by $a^2 v$. This gives

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}.$$

The left-hand side of the equality depends only upon t and the right-hand one is independent of t. It follows that

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. Thus $v(t, x) = T(t)X(x)$ is a solution of the heat equation that satisfies boundary conditions (3) if and only if T and X satisfy the ordinary differential equation

$$T'(t) + \lambda a^2 T(t) = 0, \qquad t \in (0, +\infty) \tag{5}$$

and the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad x \in (0, L), \tag{6}$$

$$X'(0) = 0, \quad X'(L) = 0. \tag{7}$$

respectively.

The Sturm-Liouville problem (6)–(7) always admits the trivial solution $X = 0$, but this is of no use to us. We must find all values of λ

for which there exist non-trivial solutions of (6)–(7). It is possible for non-negative λ only. In fact, if $\lambda < 0$, then equation (6) has the general solution

$$X(x) = C_1 \sinh(\sqrt{-\lambda}x) + C_2 \cosh(\sqrt{-\lambda}x).$$

From boundary conditions (7) we have

$$C_1 = 0, \quad C_2 \sinh(\sqrt{-\lambda}\, L) = 0.$$

Therefore $\sinh(\sqrt{-\lambda}\, L) = 0$, which is impossible for the positive number $\sqrt{-\lambda}\, L$.

If $\lambda = 0$, then there exists a nonzero solution $X_0 = 1$ of (6), (7). For $\lambda > 0$ we have $X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. Substituting X into (7) yields

$$\begin{aligned}
&X'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x;\\
&X'(0) = 0 \quad \Rightarrow \quad C_2 = 0;\\
&X'(L) = 0 \quad \Rightarrow \quad C_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sin\sqrt{\lambda}L = 0,
\end{aligned}$$

since both the constants C_1 and C_2 cannot be zero simultaneously. Then X need not be identically zero if and only if $\sin\sqrt{\lambda}L = 0$, that is, if

$$\lambda_n = \frac{\pi^2 n^2}{L^2}, \qquad n = 1, 2, \dots .$$

These values are called the eigenvalues of the problem. The corresponding solutions are

$$X_n(x) = \cos\frac{\pi n x}{L}, \qquad n = 1, 2, \dots .$$

Next, we can solve equation (5) for all eigenvalues:

$$\begin{aligned}
&T_0' = 0 \quad \Rightarrow \quad T_0(t) = A_0;\\
&T_n' + \frac{a^2\pi^2 n^2}{L^2} T_n = 0 \quad \Rightarrow \quad T_n(t) = A_n e^{-\frac{a^2\pi^2 n^2}{L^2}t}.
\end{aligned}$$

We attempt to represent the solution u of (1)–(3) as an infinite series

$$u(t,x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{a^2\pi^2 n^2}{L^2}t} \cos\frac{\pi n x}{L}.$$

We need to determine the coefficients A_n in such a way that initial condition (2) holds. We have

$$u(0,x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{\pi n x}{L} = \sin\frac{\pi x}{L}.$$

The last series is called a Fourier series of function $\sin\frac{\pi x}{L}$. Moreover,

$$A_0 = \frac{1}{L}\int_0^L \sin\frac{\pi x}{L}\,dx, \qquad A_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi n x}{L}\,dx.$$

$$A_0 = \int_0^L \sin\frac{\pi x}{L}dx = \frac{L}{\pi}\int_0^L \sin\frac{\pi x}{L}d\left(\frac{\pi x}{L}\right) = -\frac{L}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2L}{\pi}.$$

$$A_n = \int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi n x}{L}dx = \frac{1}{2}\int_0^L\left(\sin\frac{\pi(n+1)x}{L} + \sin\frac{\pi(n-1)x}{L}\right)dx$$

$$= \frac{1}{2}\left(\frac{L}{\pi(n+1)}\int_0^L \sin\frac{\pi(n+1)x}{L}d\left(\frac{\pi(n+1)x}{L}\right)\right.$$

$$\left.+\frac{L}{\pi(n-1)}\int_0^L \sin\frac{\pi(n-1)x}{L}d\left(\frac{\pi(n-1)x}{L}\right)\right)$$

$$= \frac{1}{2}\left(-\frac{L}{\pi(n+1)}\cos\frac{\pi(n+1)x}{L}\Big|_0^L - \frac{L}{\pi(n-1)}\cos\frac{\pi(n-1)x}{L}\Big|_0^L\right)$$

$$= (1-(-1)^{n+1})\left(\frac{L}{\pi(n+1)} + \frac{L}{\pi(n-1)}\right) = \frac{2(1-(-1)^{n+1})nL}{\pi(n^2-1)}$$

$$= \begin{cases} \frac{8kL}{\pi(4k^2-1)} & \text{if} \quad n = 2k \\ 0 \quad \text{if} \quad n = 2k-1 \end{cases}.$$

Answer:

$$u(t,x) = \frac{2L}{\pi} + \sum_{k=1}^{\infty}\frac{8kL}{\pi(4k^2-1)}e^{-\frac{4a^2\pi^2k^2}{L^2}t}\cos\frac{2\pi k x}{L}.$$