Answer on Question #68974 – Math – Differential Equations

Question

Solve the following differential equation
i)

$$x^2p + y^2q = (x+y)z$$

Solution

Let

$$p = \frac{\partial z}{\partial x}; \quad q = \frac{\partial z}{\partial y}.$$

We can present this quasilinear partial differential equation of the first order

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z \quad (1)$$

as

$$V(x,y,z) = 0,$$

where

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial z}},$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial V}{\partial y}}{\frac{\partial V}{\partial z}}.$$

Then

$$x^2\cdot\left(-\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial z}}\right) + y^2\cdot\left(-\frac{\frac{\partial V}{\partial y}}{\frac{\partial V}{\partial z}}\right) = (x+y)z,$$

$$-x^2\frac{\partial V}{\partial x} - y^2\frac{\partial V}{\partial y} - (x+y)z\frac{\partial V}{\partial z} = 0,$$

$$x^2\frac{\partial V}{\partial x} + y^2\frac{\partial V}{\partial y} + (x+y)z\frac{\partial V}{\partial z} = 0. \quad (2)$$

The associated system of equations is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \qquad (3)$$

If $\frac{dx}{x^2} = \frac{dy}{y^2}$, then $-\frac{1}{x} = -\frac{1}{y} + C_1$ and $u(x,y) = C_1 = \frac{1}{y} - \frac{1}{x}$, hence $y = \frac{1}{C_1 + \frac{1}{x}}$,

$$y = \frac{x}{C_1x+1} \quad (4)$$

If

$$\frac{dx}{x^2} = \frac{dz}{(x+y)z},$$

substituting for y from (4) into (3) one gets

$$\frac{dx}{x^2} = \frac{dz}{(x+\frac{x}{C_1x+1})z},$$

$$\frac{dx}{x} = \frac{dz}{\left(1+\frac{1}{C_1x+1}\right)z},$$

$$\frac{dx}{x} = \frac{(C_1x+1)}{C_1x+2} \cdot \frac{dz}{z},$$

$$\frac{C_1x+2}{x(C_1x+1)} dx = \frac{dz}{z},$$

$$\int \frac{C_1x+2}{x(C_1x+1)} dx = \int \frac{dz}{z},$$

$$\int \left(\frac{2}{x} - \frac{C_1}{C_1x+1}\right) dx = \ln|z| + \ln|C_2|,$$

$$2\ln|x| - \ln|C_1x+1| = \ln|C_2z|,$$

$$\frac{x^2}{C_1x+1} = C_2z,$$

$$C_2 = \frac{x^2}{z(C_1x+1)},$$

$$C_2 = \frac{x^2}{z\left(\left(\frac{1}{y} - \frac{1}{x}\right)x + 1\right)}$$

$$C_2 = \frac{x^2}{z\frac{x}{y}},$$

$$v(x,y) = C_2 = \frac{xy}{z}.$$

The general integral is given by

$$F(u,v) = 0,$$

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{xy}{z}\right) = 0,$$

$$F\left(\frac{x-y}{xy}, \frac{xy}{z}\right) = 0,$$

where F is an arbitrary function.

For example, if we take

$F(u,v) = -Cuv - 1 = -C \cdot \frac{x-y}{xy} \cdot \frac{xy}{z} - 1 = C \cdot \frac{y-x}{z} - 1 = 0,$

then $z = C(y-x)$ is one of solutions of (1).

Function $z = 0$ is also a solution of equation (1) because $\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0.$

If $x = 0$, then $y^2 \frac{\partial z}{\partial y} = yz, \quad \frac{dz}{z} = \frac{dy}{y}, \quad \frac{z}{y} = C_1$ and a solution of (1) is

$$(x,y,z) = (0, y, C_1y) = s(0,1,C_1).$$

If $y = 0$, then $x^2 \frac{\partial z}{\partial x} = xz, \quad \frac{dz}{z} = \frac{dx}{x}, \quad \frac{z}{x} = C_2$ and a solution of (1) is

$$(x,y,z) = (x, 0, C_2x) = t(1,0,C_2).$$

If $y = -x$, then z is arbitrary and a solution of (1) is $(x,y,z) = (x,-x,z) = (s,-s,t)$.

Answer: $F\left(\frac{x-y}{xy}, \frac{xy}{z}\right) = 0;\ z = C_1y,\ x = 0;\ z = C_2x,\ y = 0;\ y = -x.$

Question

Solve the following differential equation

ii)

$$\sqrt{p} - \sqrt{q} + 3x = 0 \quad (5)$$

Solution

$$F(x,y,z,p,q) = \sqrt{p} - \sqrt{q} + 3x,$$

$$\frac{\partial F}{\partial x} = 3, \ \ \frac{\partial F}{\partial y} = 0, \ \ \frac{\partial F}{\partial z} = 0, \ \ \frac{\partial F}{\partial p} = \frac{1}{2\sqrt{p}}, \ \ \frac{\partial F}{\partial q} = -\frac{1}{2\sqrt{q}}.$$

The characteristic system of ordinary differential equations is

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = -\frac{dp}{F_x + pF_z} = -\frac{dq}{F_y + qF_z}$$

$$\frac{dx}{\frac{1}{2\sqrt{p}}} = \frac{dy}{-\frac{1}{2\sqrt{q}}} = \frac{dz}{p \cdot \frac{1}{2\sqrt{p}} + q \cdot \left(-\frac{1}{2\sqrt{q}}\right)} = -\frac{dp}{3} = -\frac{dq}{0}$$

$$\frac{dx}{\frac{1}{2\sqrt{p}}} = \frac{dy}{-\frac{1}{2\sqrt{q}}} = \frac{dz}{\frac{\sqrt{p}}{2} - \frac{\sqrt{q}}{2}} = -\frac{dp}{3} = -\frac{dq}{0}$$

If

$$\frac{dy}{-\frac{1}{2\sqrt{q}}} = -\frac{dq}{0},$$

$$2\sqrt{q}dy = \frac{dq}{0},$$

$$\frac{dq}{\sqrt{q}} = 0dy,$$

$$2\sqrt{q} = K,$$

$$\frac{\partial z}{\partial y} = q = C_1^2 \quad (6)$$

(K, C_1 are arbitrary constants),
hence

$$z = C_1^2 y + \varphi(x),$$

$$p = \frac{\partial z}{\partial x} = \varphi'(x) \quad (7)$$

Substituting (6), (7) into equation (5)

$$\sqrt{\varphi'(x)} - C_1 + 3x = 0,$$

$$\sqrt{\varphi'(x)} = -3x + C_1$$

$$\varphi'(x) = (-3x + C_1)^2 = 9x^2 - 6xC_1 + C_1^2$$

$$\varphi(x) = 9\frac{x^3}{3} - 6C_1\frac{x^2}{2} + C_1^2 x + C_2 = 3x^3 - 3C_1x^2 + C_1^2 x + C_2,$$

where C_2 is arbitrary constant.

Then

$$z = C_1^2 y + \varphi(x) = 3x^3 - 3C_1 x^2 + C_1^2 x + C_2 + C_1^2 y \quad (8)$$

If $\frac{\partial z}{\partial x} = p = 0$, then it follows from (5) that

$\frac{\partial z}{\partial y} = q = 9x^2 \implies z = 9x^2 y + \varphi(x) = 9C_1^2 y + C_2$ (formula (8) already contains this solution).

If $\frac{\partial z}{\partial y} = q = 0$, then it follows from (5) that

$\frac{\partial z}{\partial x} = p = 9x^2 \implies z = 3x^3 + \psi(y) = 3x^3 + c$ (formula (8) already contains this solution).

Answer: $z = 3x^3 - 3C_1 x^2 + C_1^2 x + C_2 + C_1^2 y$.