Answer on Question 68963 - Math - Differential Equations

Solve, using the method of variation of parameters $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$.

Solution:

Let us first solve the corresponding homogenous linear differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

The characteristic equation $\lambda^2 - 1 = 0$ has two roots $\lambda_1 = -1$ and $\lambda_2 = 1$. Consequently, the pair of functions e^{-x} and e^x is a fundamental system of solutions and therefore the general solution has the form

$$y = C_1 e^{-x} + C_2 e^x,$$

where C_1 and C_2 are arbitrary real constants.

By the method of variation of parameters, we look for a partial solution of the non-homogenous equation in the form

$$y_* = \alpha_1(x)e^{-x} + \alpha_2(x)e^x \qquad (1)$$

with unknown functions α_1 and α_2. The derivatives of α_1, α_2 can be found as a solution of the system

$$\begin{pmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{pmatrix} \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{1+e^x} \end{pmatrix}.$$

Using Cramer's rule we solve the system

$$\Delta(x) = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = e^{-x}e^x + e^{-x}e^x = 2;$$

$$\Delta_1(x) = \begin{vmatrix} 0 & e^x \\ \frac{2}{1+e^x} & e^x \end{vmatrix} = -\frac{2e^x}{1+e^x},$$

$$\Delta_2(x) = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{2}{1+e^x} \end{vmatrix} = \frac{2e^{-x}}{1+e^x};$$

$$\alpha_1'(x) = \frac{\Delta_1(x)}{\Delta(x)} = -\frac{e^x}{1+e^x}, \quad \alpha_2'(x) = \frac{\Delta_2(x)}{\Delta(x)} = \frac{e^{-x}}{1+e^x}.$$

Then we have

$$\alpha_1(x) = -\int \frac{e^x}{1+e^x}\,dx = \begin{bmatrix} e^x = t \\ x = \ln t \\ dx = \frac{1}{t}dt \end{bmatrix} = -\int \frac{dt}{1+t} = -\ln|t+1|$$

$$= -\ln(1+e^x).$$

Next,

$$\alpha_2(x) = \int \frac{e^{-x}}{1+e^x}dx = -\int \frac{de^{-x}}{1+e^x} = \left[e^{-x} = t\right] = -\int \frac{dt}{1+\frac{1}{t}}$$

$$= -\int \frac{tdt}{t+1} = -\int \left(1 - \frac{1}{t+1}\right) dt$$

$$= -t + \ln|t+1| = -e^{-x} + \ln|e^{-x}+1| = -e^{-x} + \ln(1+e^x) - x.$$

Substituting α_1, α_2 into (1) gives the partial solution of the non-homogenous equation

$$y_* = (e^x - e^{-x})\ln(1+e^x) - xe^x - 1.$$

Finally we have the general solution of the non-homogenous equation

$$y = C_1e^{-x} + C_2e^x + (e^x - e^{-x})\ln(1+e^x) - xe^x - 1.$$

Answer: $y = C_1e^{-x} + C_2e^x + (e^x - e^{-x})\ln(1+e^x) - xe^x - 1.$