Answer on Question #68322 – Math – Differential Equations

Question

(1) Obtain the solution for the initial value problem dy/dx+(cotx)y=xcscx,y(pi/2)=1

(2) Solve the differential equation dy/dx=x(1+y^2)

(3) Solve the differential equation dy/dx+(cotx)y=xcscx

Solution

Q1. We have the initial value problem

$$\frac{dy}{dx}+(\cot x)y=x\csc x \quad y\left(\frac{\pi}{2}\right)=1$$

First we find the general solution of the equation, i.e. solve the problem

$$\frac{dy}{dx}+(\cot x)y=x\csc x$$

This equation is a linear equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where

$$P(x)=\cot x,\ \ Q(x)=x\csc x$$

To solve this differential equation, we need to multiply both sides by the integrating factor $I(x)$, where

$$|I(x)|=e^{\int P(x)dx}=e^{\int \cot x dx}$$

and integrate both sides.

Find integral

$$\int \cot x dx=\int \frac{\cos x}{\sin x}dx=\int \frac{d(\sin x)}{\sin x}=\ln|\sin x|+C$$

We are looking for a particular integrating factor, so we take C=0. Then we get

$$|I(x)|=e^{\ln|\sin x|}=|\sin x|$$

Multiply both sides of the equation by $I(x)=\sin x$

$$\sin x\frac{dy}{dx}+(\sin x\cdot\cot x)y=x(\csc x\cdot\sin x)$$

or

$$\sin x\frac{dy}{dx}+(\cos x)y=x$$

and so we can rewrite the equation as

$$\frac{d}{dx}(y\cdot\sin x)=x$$

Integrate both sides of this equation we get the general solution

$$\int \frac{d}{dx}(y\cdot\sin x)dx=\int x dx \quad \Rightarrow \quad y\cdot\sin x=\frac{x^2}{2}+c$$

where c is constant defined by the initial conditions.

Now we find c and solve the initial value problem. Substitute the initial condition

$$y\left(\frac{\pi}{2}\right) = 1$$

into the general solution:

$$1 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + c$$

or

$$1 = \frac{\pi^2}{8} + c \quad \Rightarrow \quad c = 1 - \frac{\pi^2}{8}$$

Then we obtain the solution for the initial value problem

$$y \cdot \sin x = \frac{x^2}{2} + 1 - \frac{\pi^2}{8}$$

or dividing both sides by $\sin x$,

$$y = \left(\frac{x^2}{2} + 1 - \frac{\pi^2}{8}\right) \csc x$$

Answer: the solution for the initial value problem is

$$y = \left(\frac{x^2}{2} + 1 - \frac{\pi^2}{8}\right) \csc x$$

Q2. We have the equation

$$\frac{dy}{dx} = x(1 + y^2)$$

This equation is the separable equation, i.e. it is a first-order differential equation in which the expression for dy/dx can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

For this problem

$$f(x) = x, \quad g(y) = (1 + y^2)$$

To solve this equation we rewrite it in the differential form

$$dy = x(1 + y^2)dx$$

or, dividing both sides by $1 + y^2$,

$$\frac{dy}{(1 + y^2)} = xdx$$

Then we integrate both sides of this equation:

$$\int \frac{dy}{(1 + y^2)} = \int xdx$$

We get

$$\arctan y = \frac{x^2}{2} + c$$

or

$$y = \tan\left(\frac{x^2}{2} + c\right),$$

where c is constant

Answer: The solution of the given differential equation is

$$y = \tan\left(\frac{x^2}{2} + c\right)$$

Q3. We have the equation

$$\frac{dy}{dx} + (\cot x)y = x\csc x$$

The general solution of this equation we obtained by solving problem 1:

$$y \cdot \sin x = \frac{x^2}{2} + c$$

or dividing both sides by $\sin x$,

$$y = \left(\frac{x^2}{2} + c\right)\csc x$$

Answer: The solution of the given differential equation is

$$y = \left(\frac{x^2}{2} + c\right)\csc x$$