Answer on Question 66694 - Math - Linear Algebra

Consider the linear operator $T : \mathbb{C}^4 \to \mathbb{C}^4$, defined by $T(z_1, z_2, z_3, z_4) = (-iz_2, iz_1, -iz_4, z_3)$.

i) Compute T^* and check whether T is selfadjoint.

ii) Check whether T is unitary.

Solution

i) For linear operator T the matrix representation is

$$\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We recall, that an operator T^* is called adjoint for the linear operator T if for all $x, y \in \mathbb{C}^4$ $(Tx, y) = (x, T^*y)$. The matrix representation for T^* can be found as

$$T^* = (\overline{T})^T = \overline{(T^T)}$$

where A^T denotes the transpose and $\overline{A}$ denotes the matrix with complex conjugated entries.

In our case

$$T^* = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \end{bmatrix}$$

and the adjoint operator $T^*(z_1, z_2, z_3, z_4) = (-iz_2, iz_1, z_4, iz_3)$. Since $T \neq T^*$, then T is not selfadjoint.

ii) We recall, that a unitary operator is a bounded linear operator on a Hilbert space that satisfies $U^*U = UU^* = I$, where U^* is the adjoint of U.

In our case

$$T \cdot T^* = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

and

$$T^* \cdot T = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, T is unitary.

Answer: i) $T^*(z_1, z_2, z_3, z_4) = (-iz_2, iz_1, z_4, iz_3)$, T is not selfadjoint; ii) T is unitary.